The Integral Effect
& the intrinsic uncertainty (randomness) in dynamical systems

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MAX-PLANCK-INSTITUT
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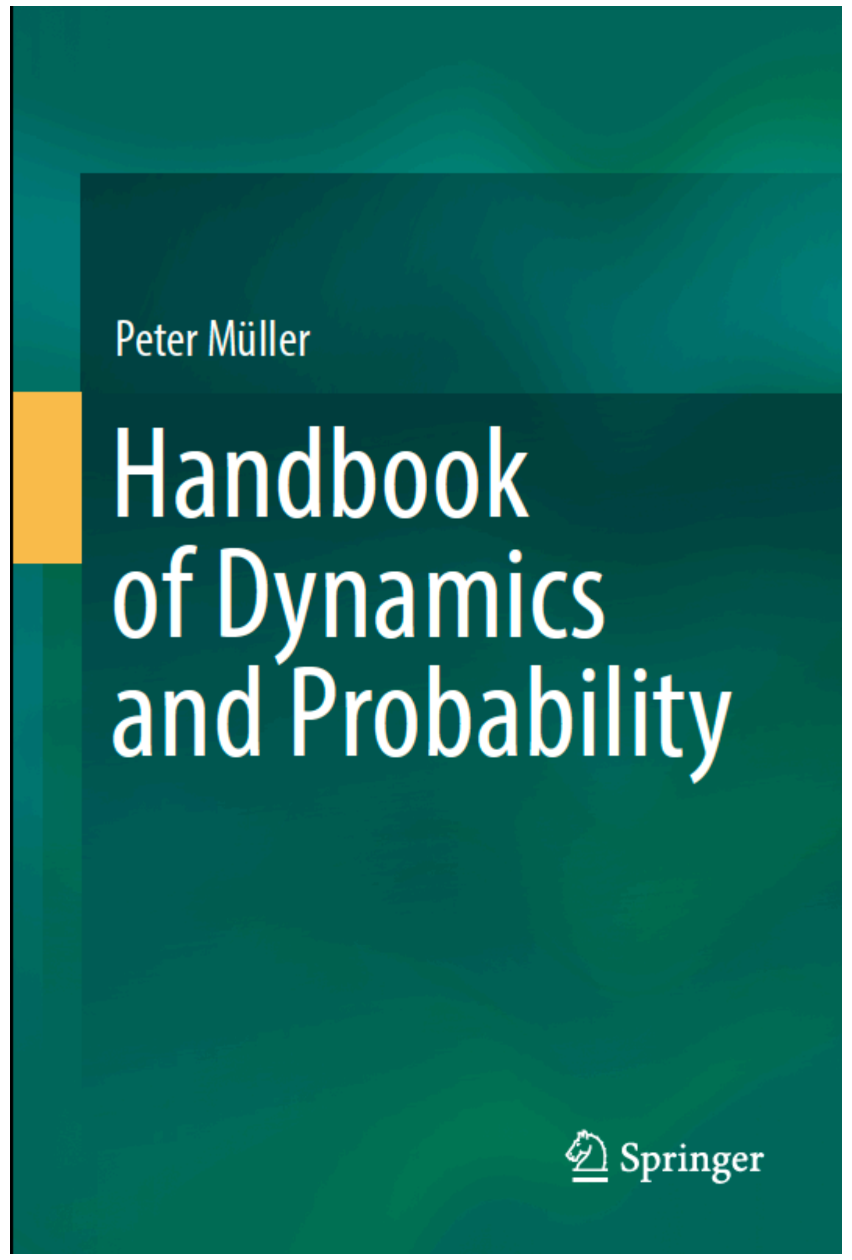
- Uncertainties arise from
  - lack of knowledge
  - inability of controlling certain things
- Intrinsic uncertainty is a phenomenon which occurs even in case of complete understanding and full control
- Physical origin of intrinsic uncertainty is unknown (apart from that at quantum level)

Peter Müller, 2022



For a dynamical system described by $\frac{dx}{dt} = f(\mathbf{x})$, randomness is

- observed in ultra low-frequency variations (ULFV) of equilibrium solution of x

ULFV not explainable by deterministic working of differential forcing f

(*On Equilibrium Fluctuations*, von Storch, 2022, Tellus)

- $(2\pi\omega)^2\ \Gamma^x(\omega) = \Gamma^f(\omega)$

- At frequency $\omega = 0$,

$$0\ \Gamma^x(0) = \Gamma^f(0)$$

- $\Gamma^f(0)$ must vanish to ensure equilibrium solution

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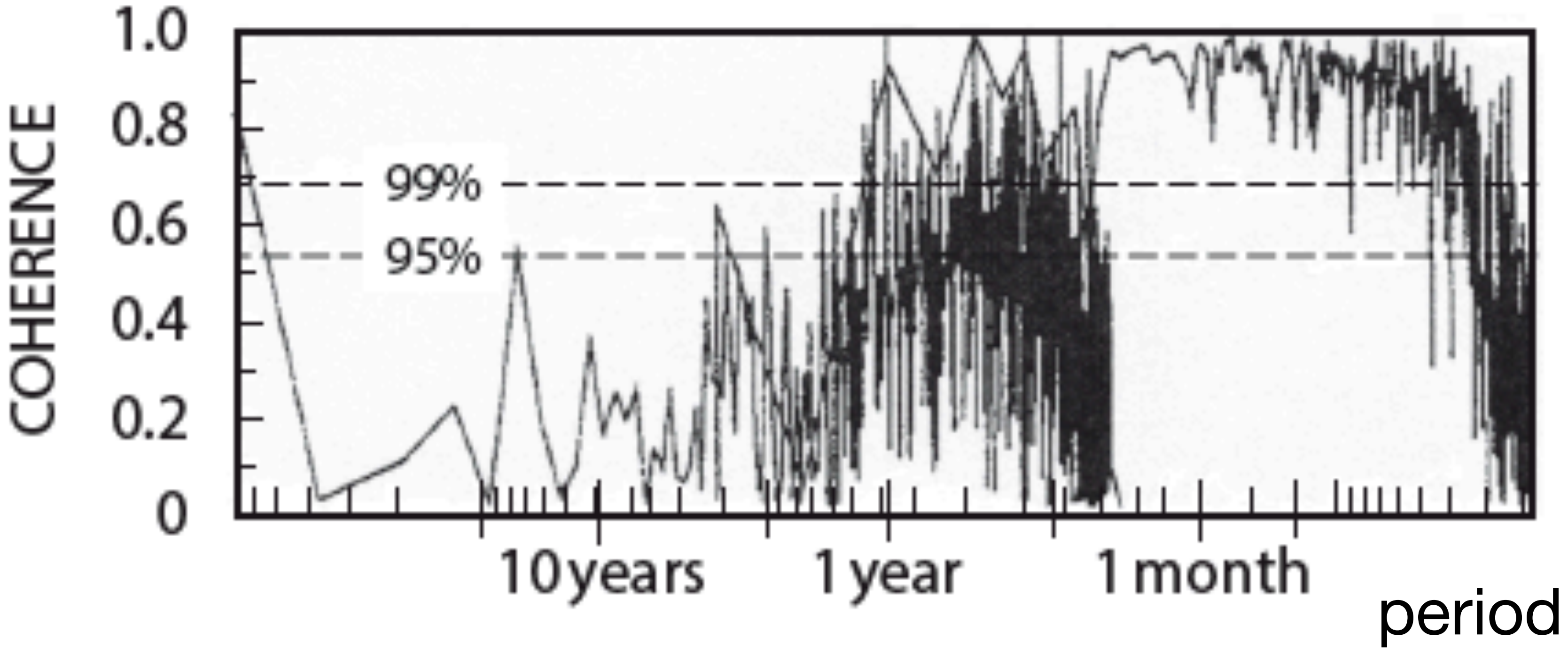
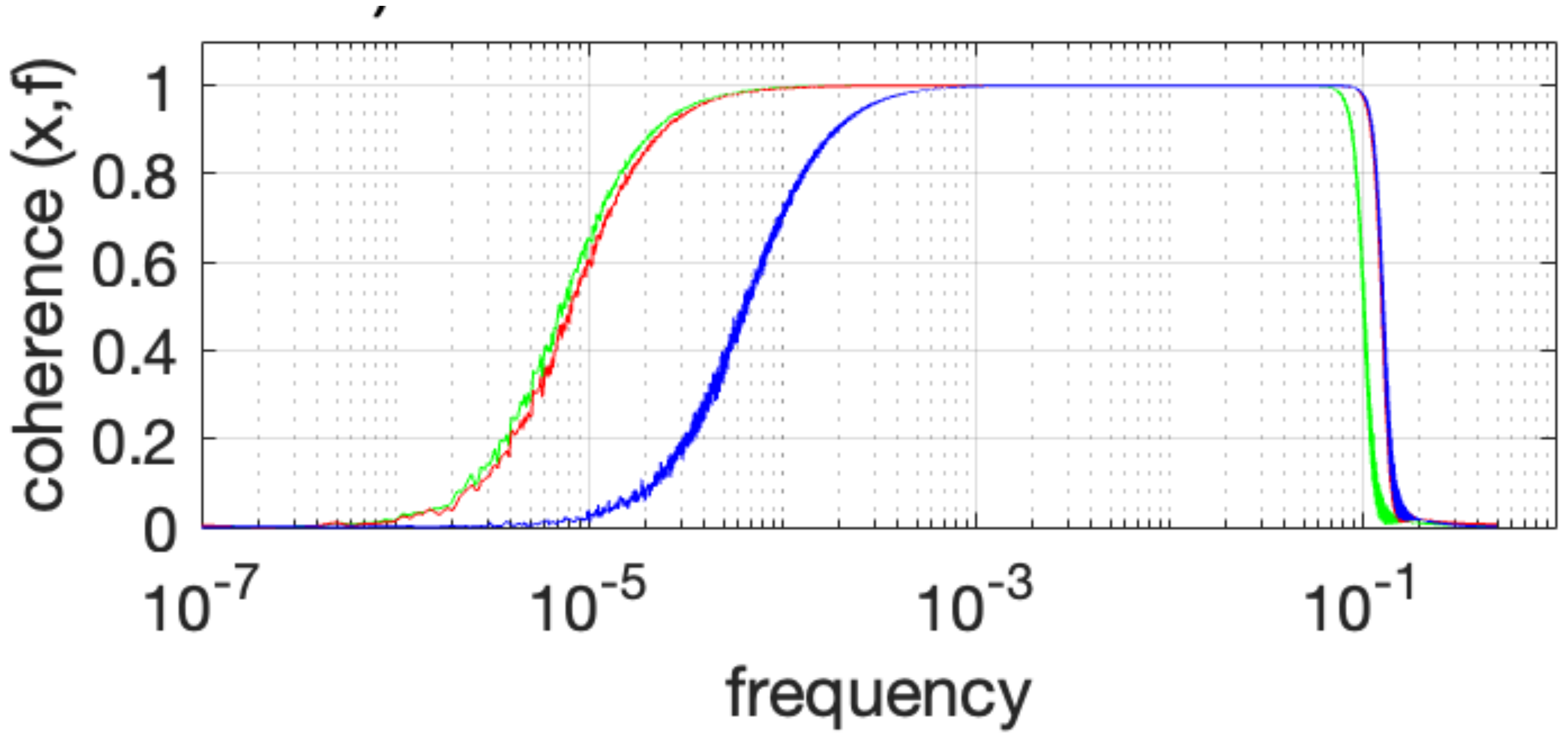


x = axial angular momentum of the atmosphere simulated by a coupled climate model
f = the respective torque

(von Storch, 1999)



x = one of the three variables of the Lorenz' 63 model
f = the dynamics of the respective variable

ULFV results from the working of internal forcing g_τ

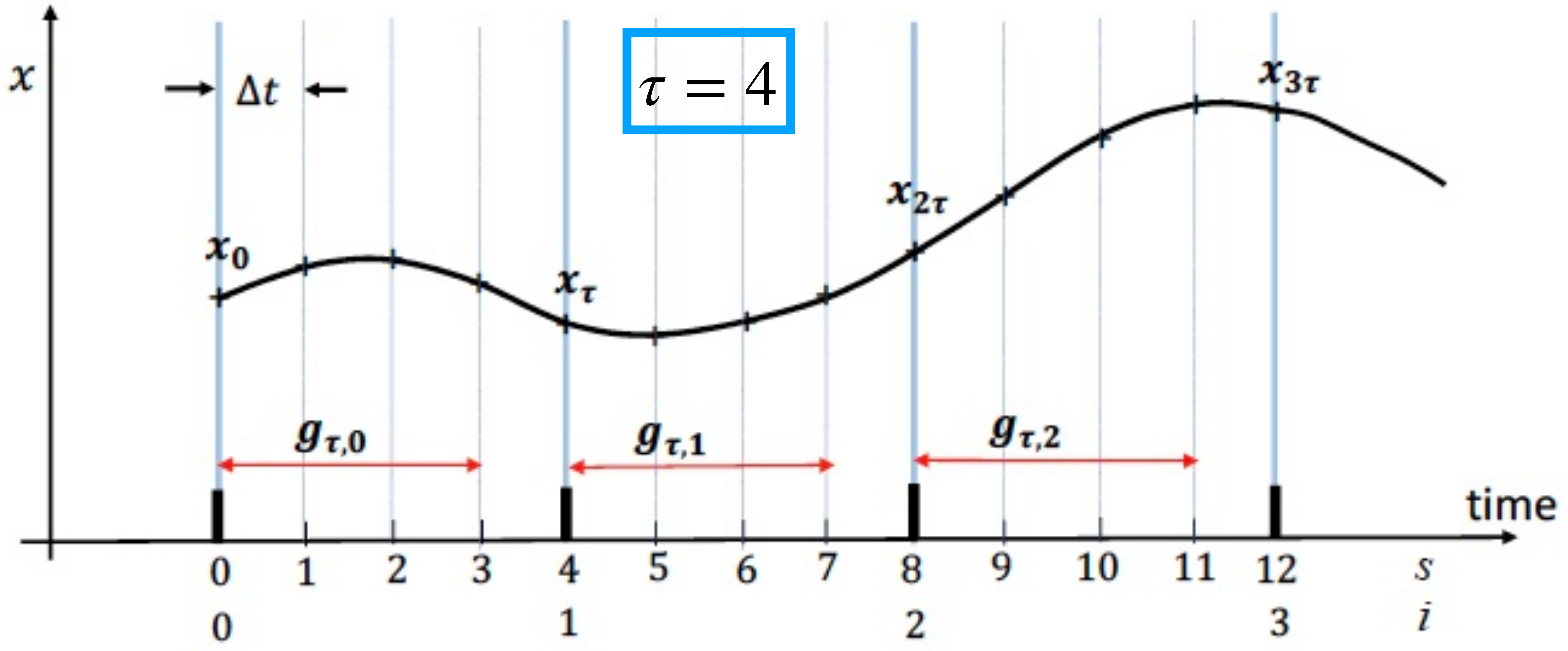
→ The physical origin of randomness

Preliminaries

- We consider only dynamical systems described by $dx/dt = f(\mathbf{x})$ that have an equilibrium solution $x(t)$, which
  - varies stationarily for ever when left alone
  - has a time independent variance - equilibrium variance
- Almost all systems of our interests do not have analytical solutions and have to be solved numerically.
- All numerical evidences are derived from the Lorenz model (1963)

The integral forcing g_τ

Discretize time axis using increment Δt. Set $\Delta t = 1$



- $\{x_s\} = \{x_s \,|\, s \in \mathbb{Z}_*\}$: solution at every time steps
- $\{x_{i\tau}\} = \{x_{i\tau} \,|\, i \in \mathbb{Z}_*\}$: solution at every τ time steps

Differential forcing f_s :

- $f_s = f(\mathbf{x}_s)$
- $x_{s+1} = x_s + f_s$

τ-stepping integral forcing $g_{\tau,i}$:

- $g_{\tau,i} = \sum_{s=i\tau}^{(i+1)\tau} f_s$, for $\tau \in \mathbb{Z}_+$
- $g_{1,i} = f_i$, for $\tau = 1$
- $x_{(i+1)\tau} = x_{i\tau} + g_{\tau,i}$

Properties of the integral forcing g_τ

An integral forcing can be written as:

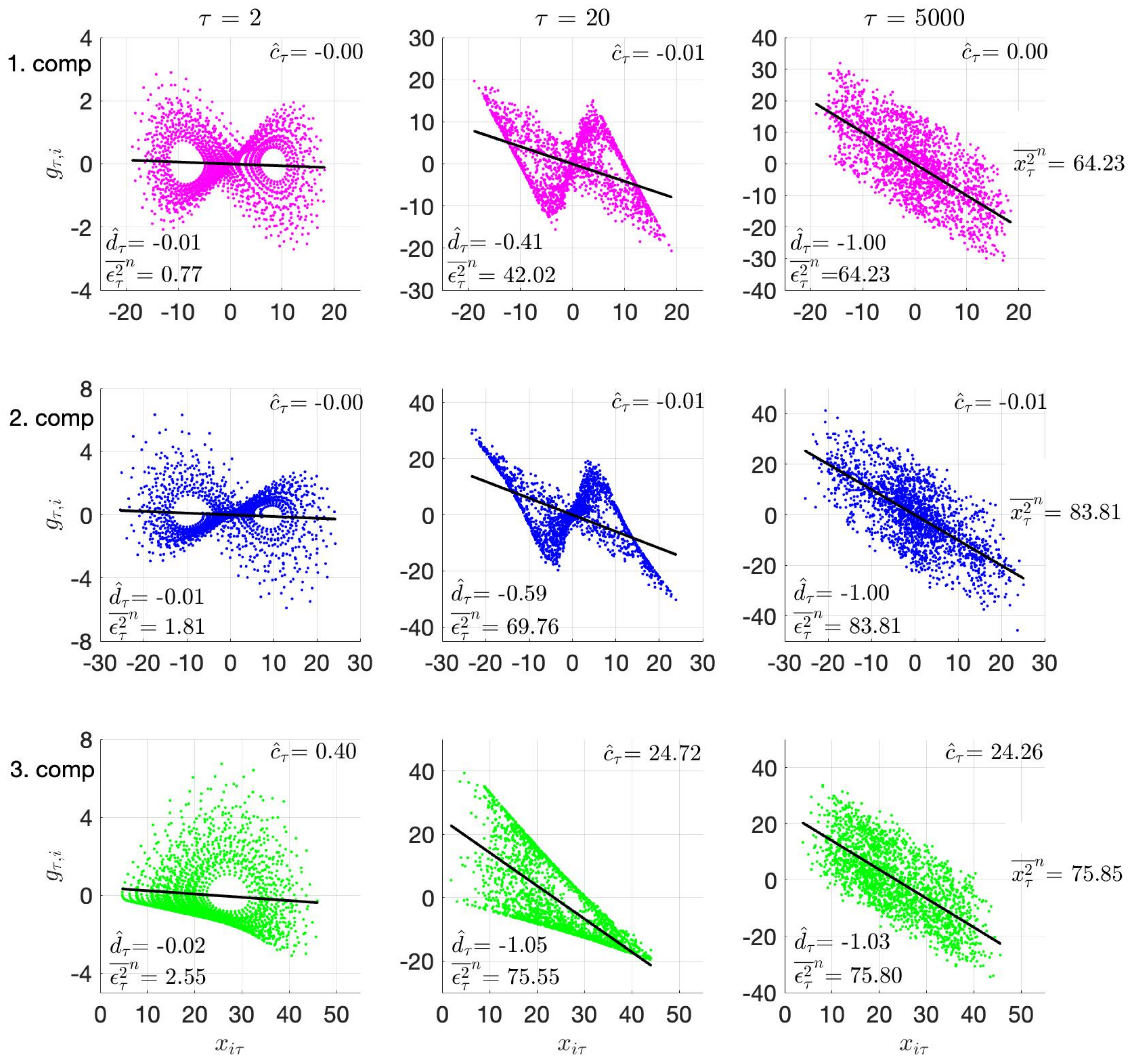
$$g_{\tau,i} = \sum_{s=i\tau}^{(i+1)\tau-1} f_s = \hat{c}_\tau + \hat{d}_\tau x_{i\tau} + \hat{\epsilon}_{\tau,i}$$

$\hat{c}_\tau$: intercept $\quad \hat{d}_\tau$: repression slope

$\hat{\epsilon}_{\tau,i}$: residual $g_{\tau,i} - \left(\hat{c}_\tau + \hat{d}_\tau x_{i\tau}\right)$

$\hat{}$: derived from n data points along an equilibrium solution, here n=10^6

- $g_{\tau,i}$ becomes increasingly linear in $x_{i\tau}$ with increasing τ
- Once $g_{\tau,i}$ is linear in $x_{i\tau}$, $\hat{\epsilon}_{\tau,i}$ behaves like a white noise



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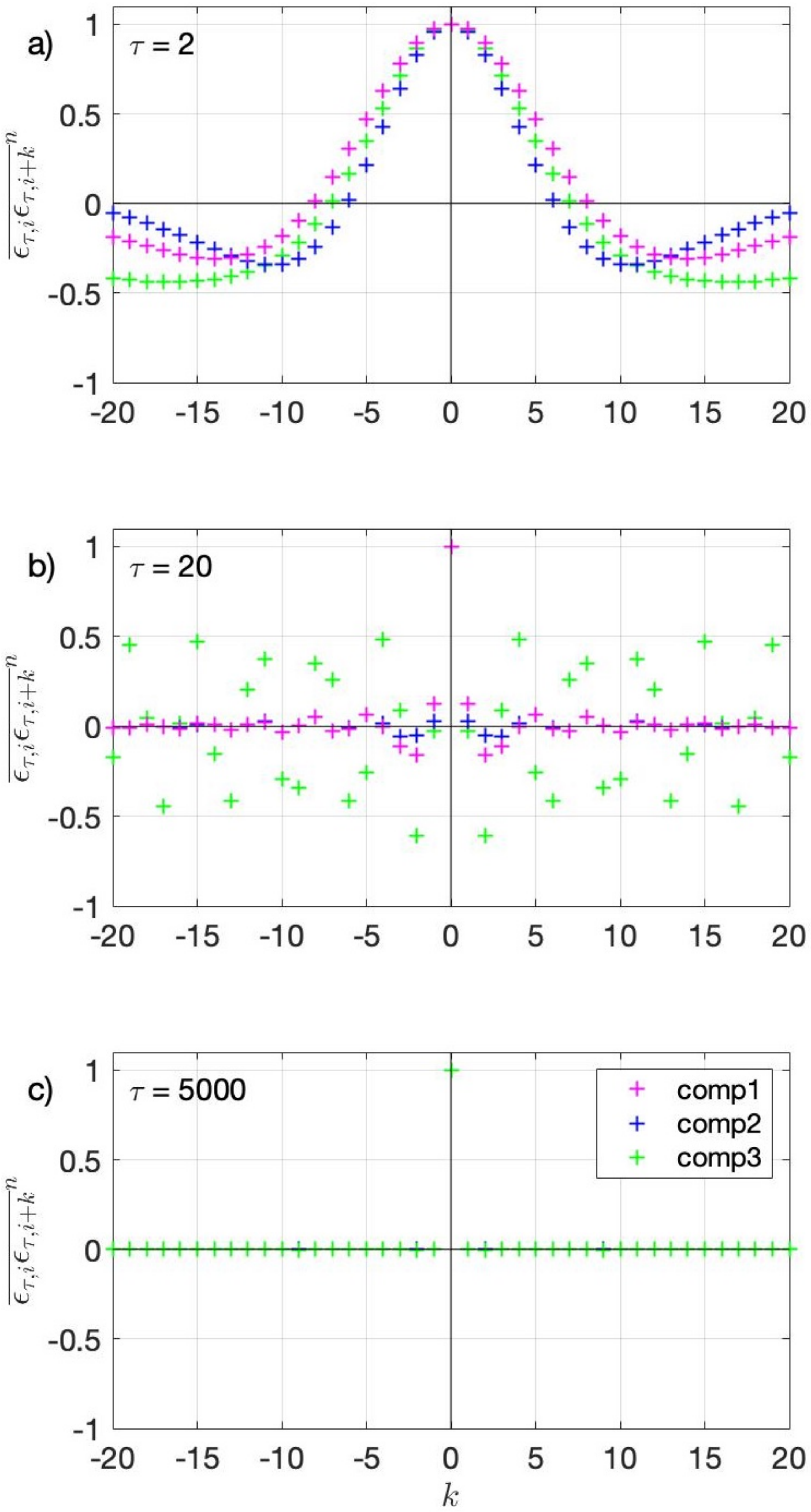
Auto-correlation function $\overline{\epsilon_{\tau,i}\epsilon_{\tau,i+k}}^n$ as a function of lag k



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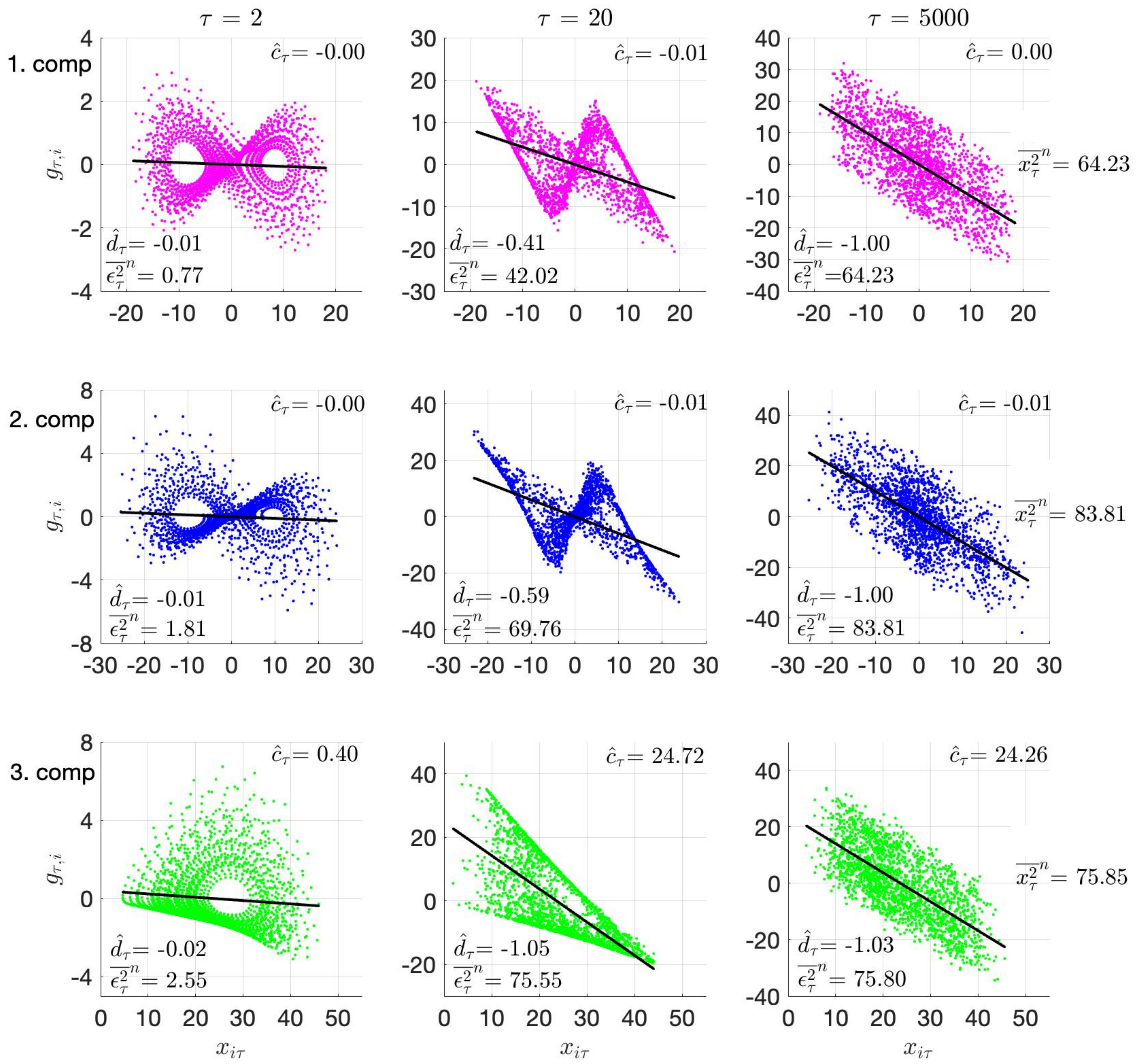
$$g_{\tau,i} = \sum_{s=i\tau}^{(i+1)\tau-1} f_s = \hat{c}_\tau + \hat{d}_\tau x_{i\tau} + \hat{\epsilon}_{\tau,i}$$

dissipating component of $g_{\tau,i}$ with strength $|d_\tau|$

fluctuating component of $g_{\tau,i}$ with strength $\overline{\epsilon_\tau^2}^n$

- $g_{\tau,i}$ becomes increasingly linear in $x_{i\tau}$ with increasing τ
- Once $g_{\tau,i}$ is linear in $x_{i\tau}$, $\hat{\epsilon}_{\tau,i}$ behaves like a white noise
- $\hat{d}_\tau$ is always negative
- $\overline{\epsilon_\tau^2}^n$ increases with $|\hat{d}_\tau|$, and stops to increases and becomes equals to $\overline{x^2}^n$ when $|\hat{d}_\tau|$=1, which happens when $\tau > \tau_0$



Properties of the integral forcing g_τ

- The dissipating and fluctuating component of g_τ are related to each other following the *FD*-curve:

$$\boxed{Var(\epsilon_\tau) = Var(x)\left(1-(1+d_\tau)^2\right)}$$

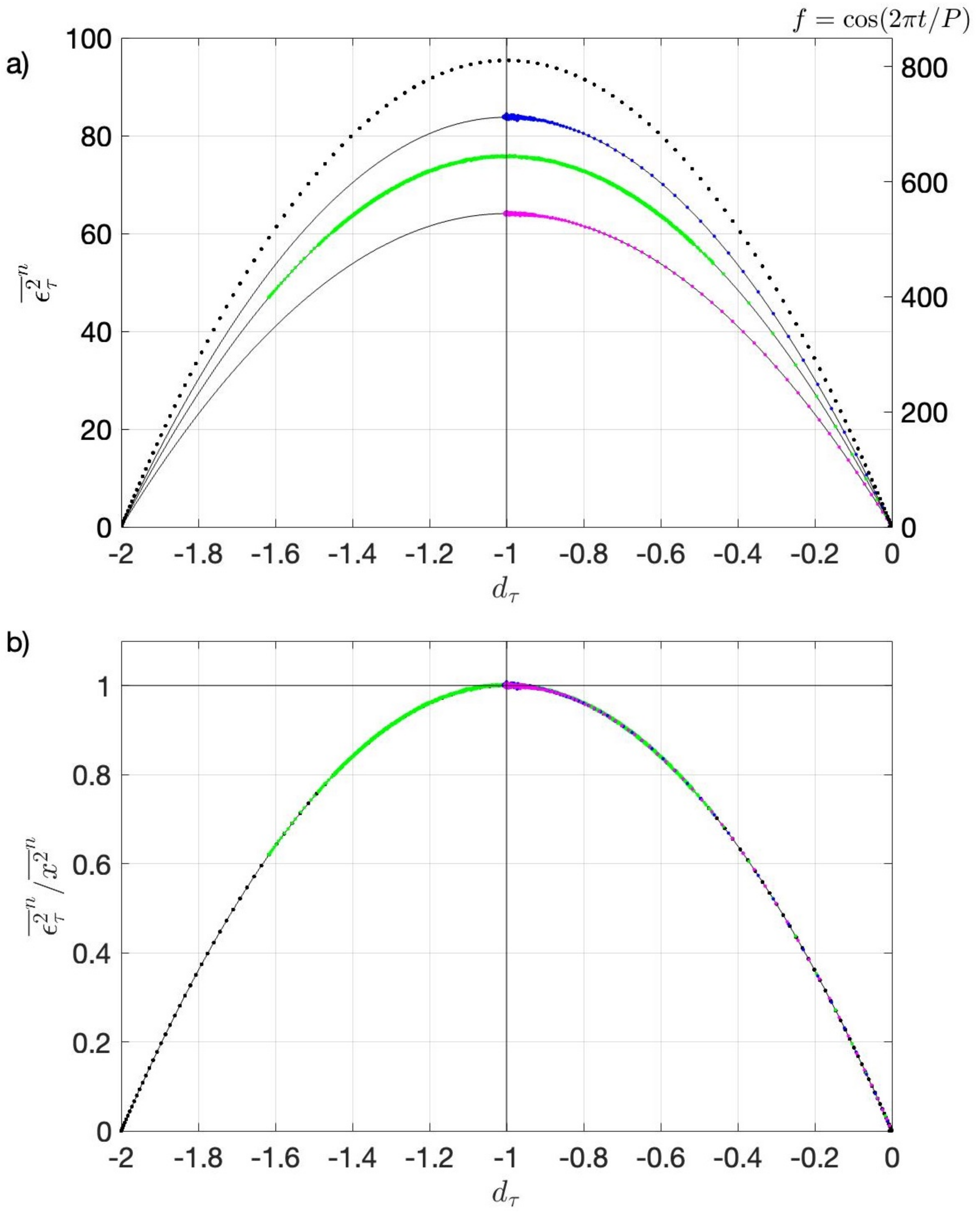
with $\quad Var(\epsilon_\tau) = \lim_{n\to\infty} \overline{\epsilon_\tau^2}^n, \qquad Var(x) = \lim_{n\to\infty} \overline{x^2}^n$

- $\left(\hat{d}_\tau, \overline{\epsilon_\tau^2}^n\right)$-points lie on the *DF*-curve
- $1 + d_\tau = \rho_\tau$ so that $d_\tau \in [-2,0]$
- $\overline{\epsilon_\tau^2}^n$ reaches its maximum at d_τ=-1, which equals $\overline{x^2}^n$
- The *FD*-curve is independent of the functional form of f
- Different f make $\left(d_\tau, \overline{\epsilon_\tau^2}^n\right)$-points to populate different parts of the DF-curve

$y = a\left(1-(1+z)^2\right)$:

$\left(\hat{d}_\tau, \overline{\epsilon_\tau^2}^n\right)$, $\tau = 1,\cdots,10^3$ from Lorenz model:

$\left(\hat{d}_\tau, \overline{\epsilon_\tau^2}^n\right)$, $\tau = 1,\cdots,10^3$ from $dx/dt = \cos(2\pi t/P)$:



The integral effect = The ability of g_τ in producing white-noise like τ-stepping solution

- Even though equivalent in determining $\{x_s\}$ at a time, the summation (needed for obtaining g_τ) makes f and g_τ to contain different amounts of information about time sequence

  $\longrightarrow$ f and g_τ generate variations in x in different ways

- f_s generates a change $x_{s+1} - x_s$ $\longrightarrow$ following the classical forcing-response relation

  Variations in f at a frequency generate variations in x at the SAME frequency

- For g_τ with $\tau > \tau_0$, we have $x_{(i+1)\tau} = c_\tau + \epsilon_{\tau,i}$, despite $x_{(i+1)\tau} - x_{i\tau} = g_{\tau,i}$

  $\{x_{i\tau}\}$ varies at all frequencies smaller than $1/\tau_0$

Since $x_{(i+1)\tau} = x_{i\tau} + g_{\tau,i} = x_{i\tau} + \left(c_\tau + d_\tau x_{i\tau} + \epsilon_{\tau,i}\right)$,

$$\boxed{x_{(i+1)\tau} = (1+d_\tau)^{i+1}x_0 + \sum_{k=0}^{i}(1+d_\tau)^k(c_\tau + \epsilon_{\tau,i-k})}$$

d_τ=0

$$\cancel{x_{(i+1)\tau} = x_0 + \sum_{k=0}^{i}(c_\tau + \epsilon_{\tau,i-k})}$$

Like random walk: No equilibium solution possible!

d_τ=-1

$$x_{(i+1)\tau} = c_\tau + \epsilon_{\tau,i}$$

Like white noise!

Classical forcing-response relation does not exists:

- It is not meaningful to talk about a change $x_{(i+1)\tau} - x_{i\tau}$, since $x_{(i+1)\tau}$ is independent of $x_{i\tau}$
- It is also not meaningful to talk about $g_{\tau,i}$ as a forcing, since $x_{(i+1)\tau}$ equals only part of $g_{\tau,i}$

$\longrightarrow$ Randomness emerges

Integral effect does not exists for all integral forcing g_τ

Since

- f is unable to generate variations at the lowest frequencies
- g_τ with $\tau > \tau_0$ exists only at frequencies $\omega < 1/\tau_0$

the integral effect can be quantified in terms of $Var(0) = 2\omega_0\Gamma(0)$ by

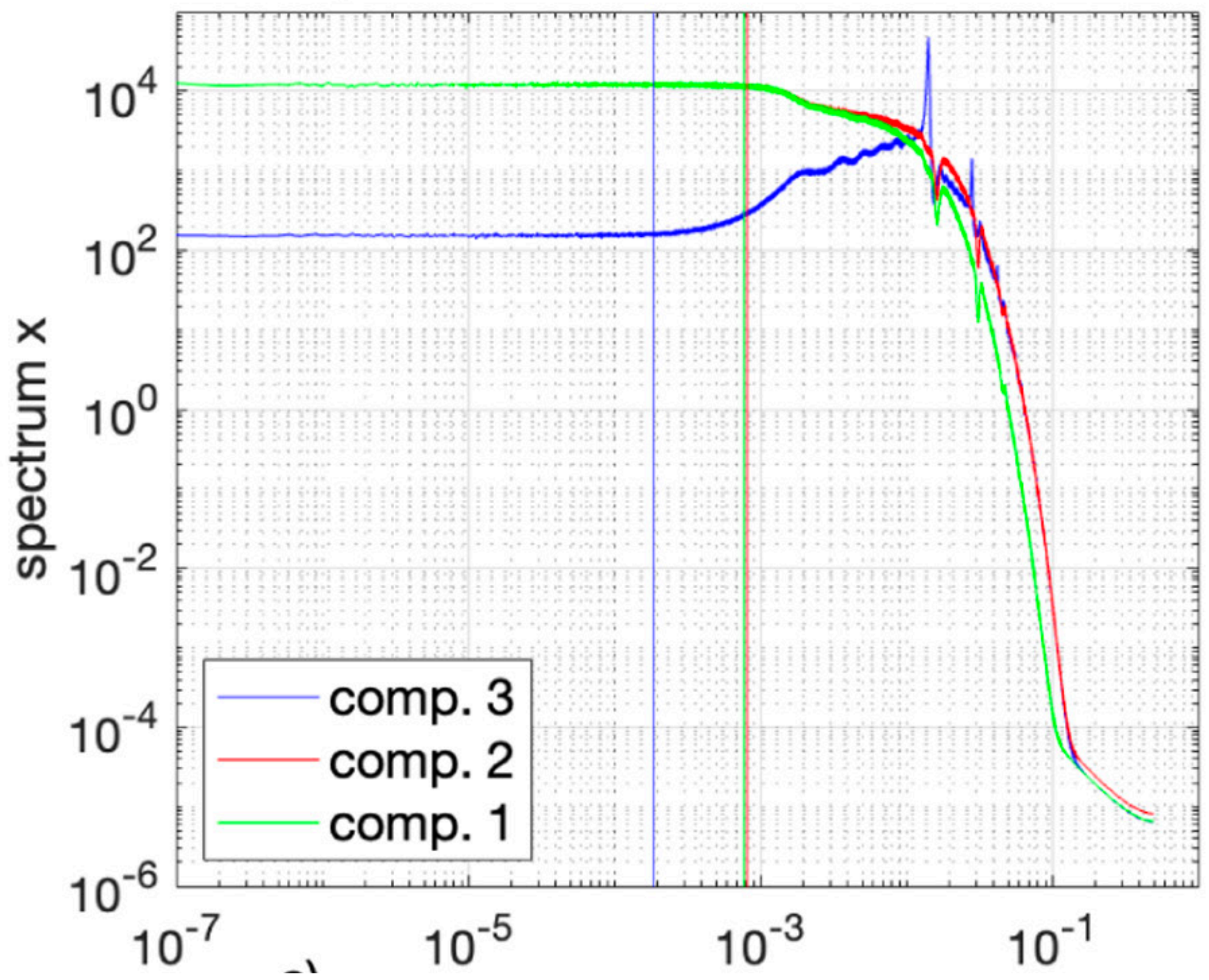
$$\frac{Var(0)}{Var(x)} = 2\omega_0\Big(2\sum_{\tau=1}^{\infty}(1+d_\tau)+1\Big)$$

where $\Gamma(0) = \sum_{\tau=-\infty}^{\infty}\gamma_\tau$, $[-\omega_0, \omega_0]$ with $\omega_0 < 1/\tau_0$ is the frequency range over which the spectrum of x has a white extension

	f^1	f^2	f^3	$\cos(2\pi t/P)$
$2\sum_{\tau=1}^{n}(1+d_\tau)+1$	86.45	68.13	1.86	-0.00
ω_0	7.03×10^{-04}	7.05×10^{-04}	7.20×10^{-05}	0
$Var(0)/Var(x)\times100\%$	12.16%	9.61%	0.03%	0.00%

The integral effect is strongest for the first two Lorenz components, but does not exist for $dx/dy = \cos(2\pi t/P)$

Lorenz model



CONCLUSIONS:

1. g_τ with $\tau \in \mathbb{Z}_+$ obeys a law-like regularity (FD-curve) that relates its dissipating component characterized by d_τ to its fluctuating component characterized by $Var(\epsilon_\tau)$, independent of the functional form of f

2. Randomness is an intrinsic feature of $dx/dt = f$ that

- results from the joint working of the dissipating and the fluctuating component of g_τ
- only „visible“ by integrating $dx/dt = f$ forward in time

3. With respect to the equilibrium variance of x, time is irreversible and has an arrow !

Thanks!