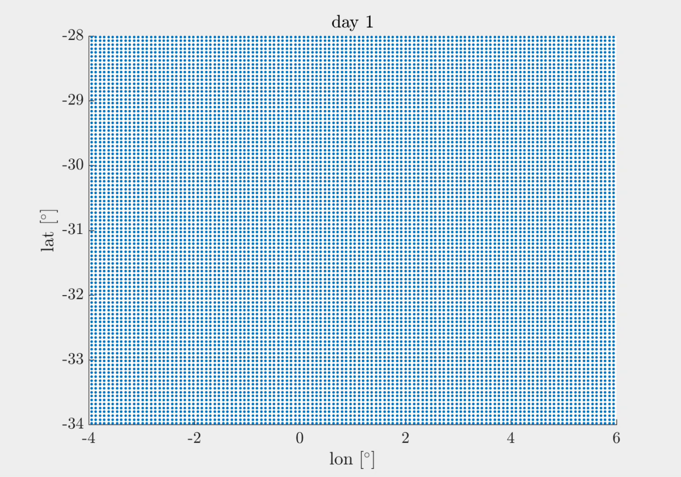
Data Based Computation of Coherent Sets in Fluids

Alvaro de Diego[1] joint work with Gary Froyland[2], Oliver Junge[1] & Peter Koltai[3]

[1]TU München [2]UNSW Sydney [3]Universität Bayreuth

March 29, 2023

Lagrangian Coherent Structures



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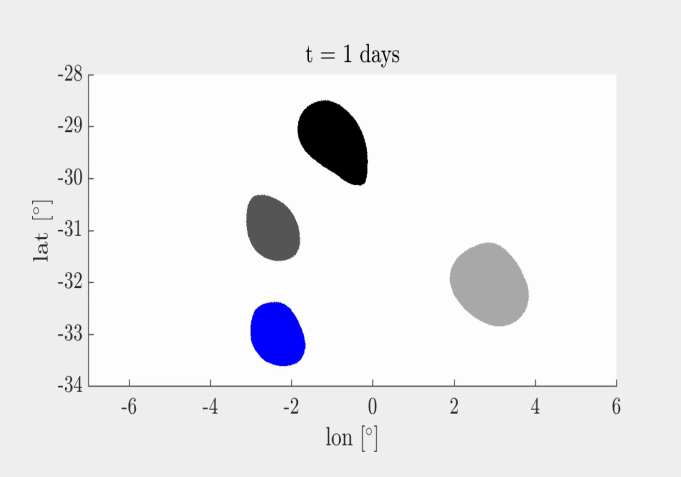

Roadmap

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- Solve the partial differential equation with only given trajectory data.

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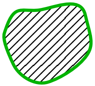
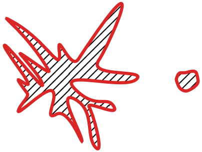


Good set D



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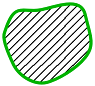

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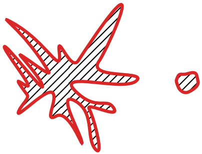

Bad sets D

Good sets $D \subset M$ make the quantity

$$\frac{\ell_{d-1}(\partial D)}{\ell_d(D)}$$

small.

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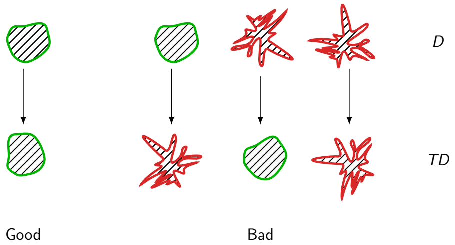

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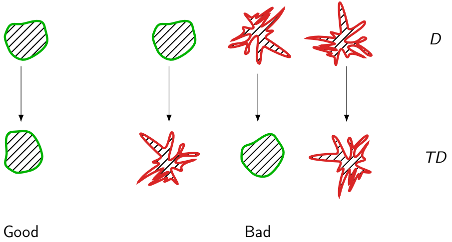

Good sets $D \subset M$ make the quantity

$$\frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(TD))}{2\ell_d(D)}$$

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The geometric problem

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Definition

Let $M \subseteq \mathbb{R}^n$ (bounded and open) and $T\colon M \to M$ volume preserving diffeomorphism. Define the *dynamic Cheeger ratio* of a set $D \subset M$ as

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Dynamic Cheeger Problem

Find a set $D \subset M$ of minimal dynamic Cheeger ratio.

An equivalent variational problem

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L^1 Variational Problem

Find $u\colon M \to \mathbb{R}$ with $u|_{\partial M} \equiv 0$ such that

$$\frac{\|\nabla u\|_1 + \|\nabla(u \circ T^{-1})\|_1}{2\|u\|_1}$$

is minimal (where $\|v\|_1 = \int_M |v|$)

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Theorem (Froyland 2015, Froyland & Junge 2019)

The minimal value of the variational problem coincides with the minimal value of the geometric problem.

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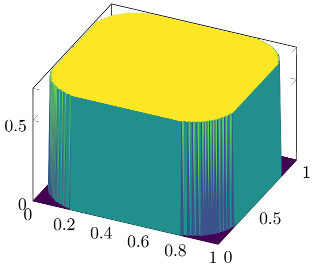

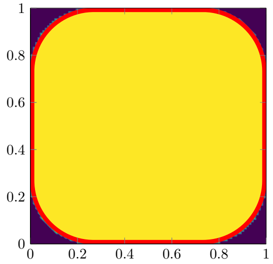

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Find $u : M \to \mathbb{R}$ with $u|_{\partial M} \equiv 0$ such that

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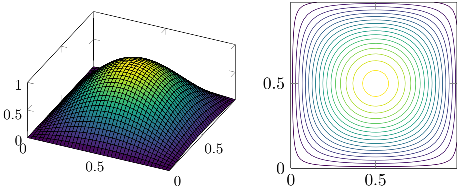

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$$\inf_{D \subseteq M} \frac{\ell_{d-1}(\partial D)}{\ell_d(D)} \leq 2\sqrt{|\lambda|},$$

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- Level sets are still good on average, even if not optimal.
- Problem much easier to solve.

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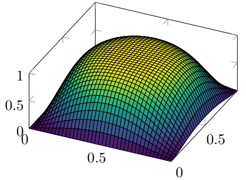

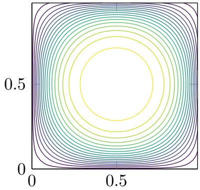

Eigenfunctions get "flatter" again, but in experiments the best level set is already near the optimum for $p = 2$.

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Theorem

Let $0 \neq u \in H_0^1(M)$ be a minimizer of the Rayleigh quotient

$$\frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

Then u is the first eigenfunction of $-\Delta$, i.e. there is a $\lambda > 0$ such that

$$-\Delta u = \lambda u$$

and λ is minimal.

The dynamic Laplacian

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Definition (Froyland 2015)

Let T^*, T_* be defined by $T^*u = u \circ T^{-1}$ and $T_*u = u \circ T$. Define the *dynamic Laplacian* $\bar{\Delta}$ by

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Theorem

The solution of the L^2 variational problem is the first Dirichlet eigenfunction of $\bar{\Delta}$, i.e. it solves

$$-\bar{\Delta} u = \lambda u \text{ on } M \,, \quad u \equiv 0 \text{ on } \partial M$$

for minimal λ.

Computing eigenfunctions

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Weak formulation (Dirichlet boundary conditions): find $u \in H_0^1(M)$ and $\lambda \in \mathbb{R}$ such that

$$\int_M \frac{1}{2}(\nabla u \cdot \nabla v + \nabla(T^* u) \cdot \nabla(T^* v)) = \lambda \int_M uv \quad \forall v \in H_0^1(M).$$

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We only want to use given trajectory data.

Collocation on adapted meshes (Froyland & Junge, 2019)

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Given x_i and $T(x_i)$ for $i \in \{1, \ldots, N\}$, calculate two triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ with the x_i and the $T(x_i)$ as vertices respectively.

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Let ϕ_k^0 and ϕ_k^1 be the piecewise linear hat functions on $\mathcal{T}_0$ and $\mathcal{T}_1$. If

$$u = \sum_{k=1}^{N} \alpha_k \varphi_k^0$$

is a piecewise linear function on $\mathcal{T}_1$ then approximate

$$T^* u \approx \sum_{k=N}^{k} \alpha_k \varphi_k^1$$

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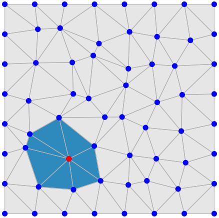

φ_k^0

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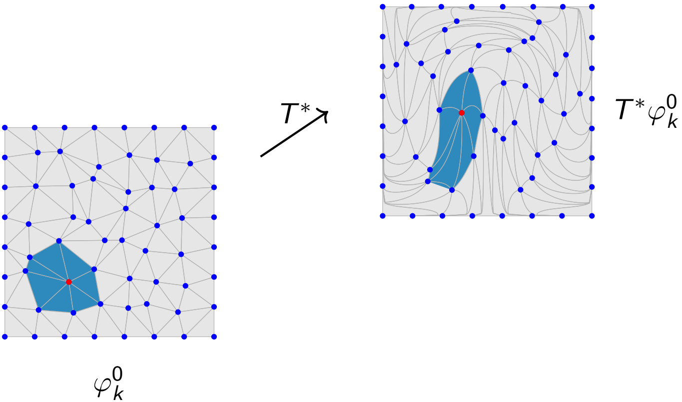

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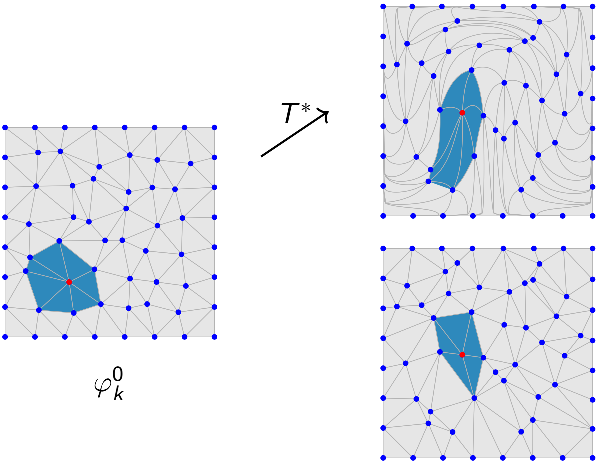

$T^* \varphi_k^0$

Approximation φ_k^1 on adapted mesh

Multiple structures

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- Different geometric problem: partition into two subsets. Measure

$$\frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(TD))}{2\min(\ell_d(D), \ell_d(M \backslash D))}.$$

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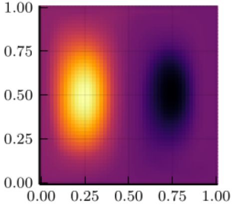
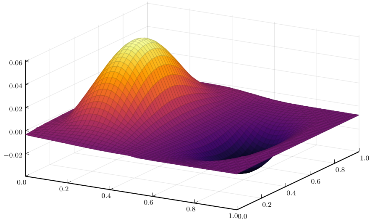
Leads to Neumann boundary conditions in PDE.





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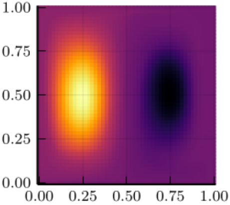

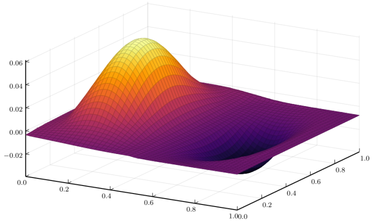

- For finding multiple structures: use higher eigenfunctions of $\bar{\Delta}$ and methods based on spectral clustering (Froyland & Dellnitz 2003, Froyland 2005).

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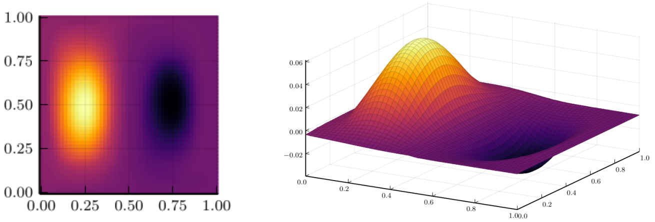

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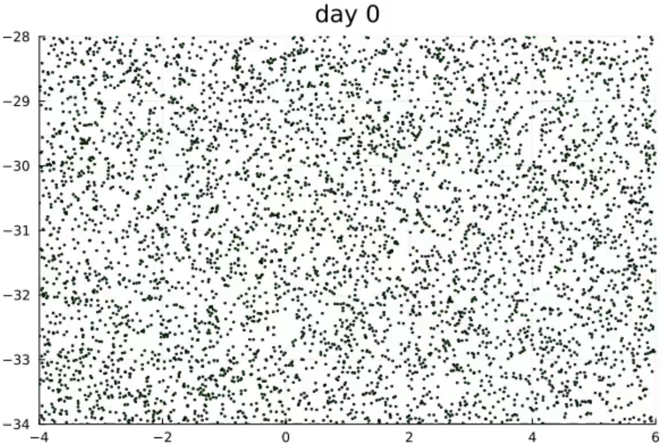
Geostrophic ocean flow (SSALTO/DUACS, distributed by AVISO).



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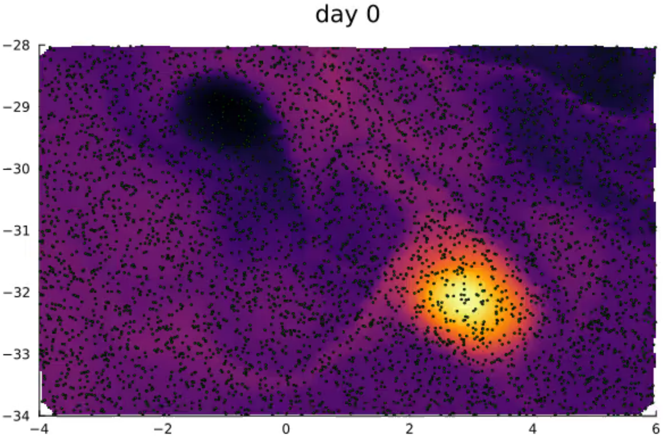
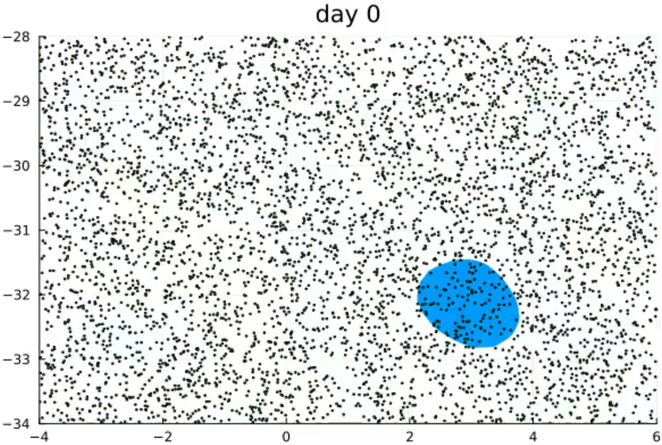
First nontrivial eigenfunction (Neumann Boundary conditions)



Best superlevel set:



Software

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- Graph Laplacian / diffusion maps based methods

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- The corresponding geometric problem can be approximated by an eigenvalue problem involving the dynamic Laplacian $\bar{\Delta}$.
- The resulting partial differential equation can be solved using only given trajectory data.

Thank you