On the Spontaneous Generation of Gravity waves

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Abstract

We investigate the spontaneous generation of gravity waves by balanced flows. It has been proven that this generation is linked to the complex poles of the balanced part of the flow. We then generate a pseudo-random sequence of poles by using the Lorenz'63 system, this last coupled to a simple finite-dimensional model that we consider. Our goal here is to construct a diagnostic tool for the emission of the unbalanced energy that takes the complex-time poles of the balanced flow as input.

On the second hand, we are implementing the "optimal balance" scheme, which is a nonlinear algorithm for the separation of gravity waves (unbalanced part of the flow) and the Rossby waves (balanced part) in the TIGAR model. The main question that we address in this part is whether the equatorial waves namely the mixed Rossby-gravity and Kelvin waves do provide a "fast" pathway in the energy transfer between Rossby and gravity waves.

- A finite-dimensional toy model, coupled with the Lorenz system is considered.

$$\dot{P} = \frac{1}{\epsilon}\Big[-JP + g\Big],$$

$$P = \begin{pmatrix} q \\ p \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ z(t) \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \gamma x - y - xz$$
$$\dot{z} = xy - \frac{8}{3}z.$$

- Fast vector field

$$w = P_{fast} = P - \sum_{i=0}^{n} \epsilon^i f_i(g), \quad f_i = -\left(J\frac{d}{dt}\right)^i Jg.$$

- Computation of fast energy

$$\|w\|^2 = \left\| P + \sum_{i=0}^{n} \epsilon^i \left(J^{i+1} \frac{d^i g}{dt^i} \right) \right\|^2 .$$

- Analytical solution and amplitude of unbalanced energy
  - Outer region: $|t - t_\star| = O(1)$

$$q(t) \sim Ae^{it/\epsilon} + Be^{-it/\epsilon} + \frac{a_{-1}}{t - t_\star} + \frac{\bar{a}_{-1}}{t - \bar{t}_\star} + \cdots$$

  - Inner region: $|t - t_\star| \leq \epsilon$

$$\phi(\tau) \sim \frac{1}{2i}\Big[e^{-i\tau}E_1(-i\tau) - e^{i\tau}E_1(i\tau)\Big],$$

$$\tau = \frac{t - t_*}{\epsilon}, \quad E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt, \qquad \mathfrak{Re}(z) \geq 0.$$

- Matched asymptotic:

$$\phi(\tau) = \frac{\epsilon}{a_{-1}} q(t)$$

We take $t = ir + s = t_\star + s.$

Fast oscillations:

$$q_{fast} \sim \frac{2\pi}{\epsilon} \mathfrak{Re}(a_{-1}e^{it/\epsilon})e^{-r/\epsilon}.$$

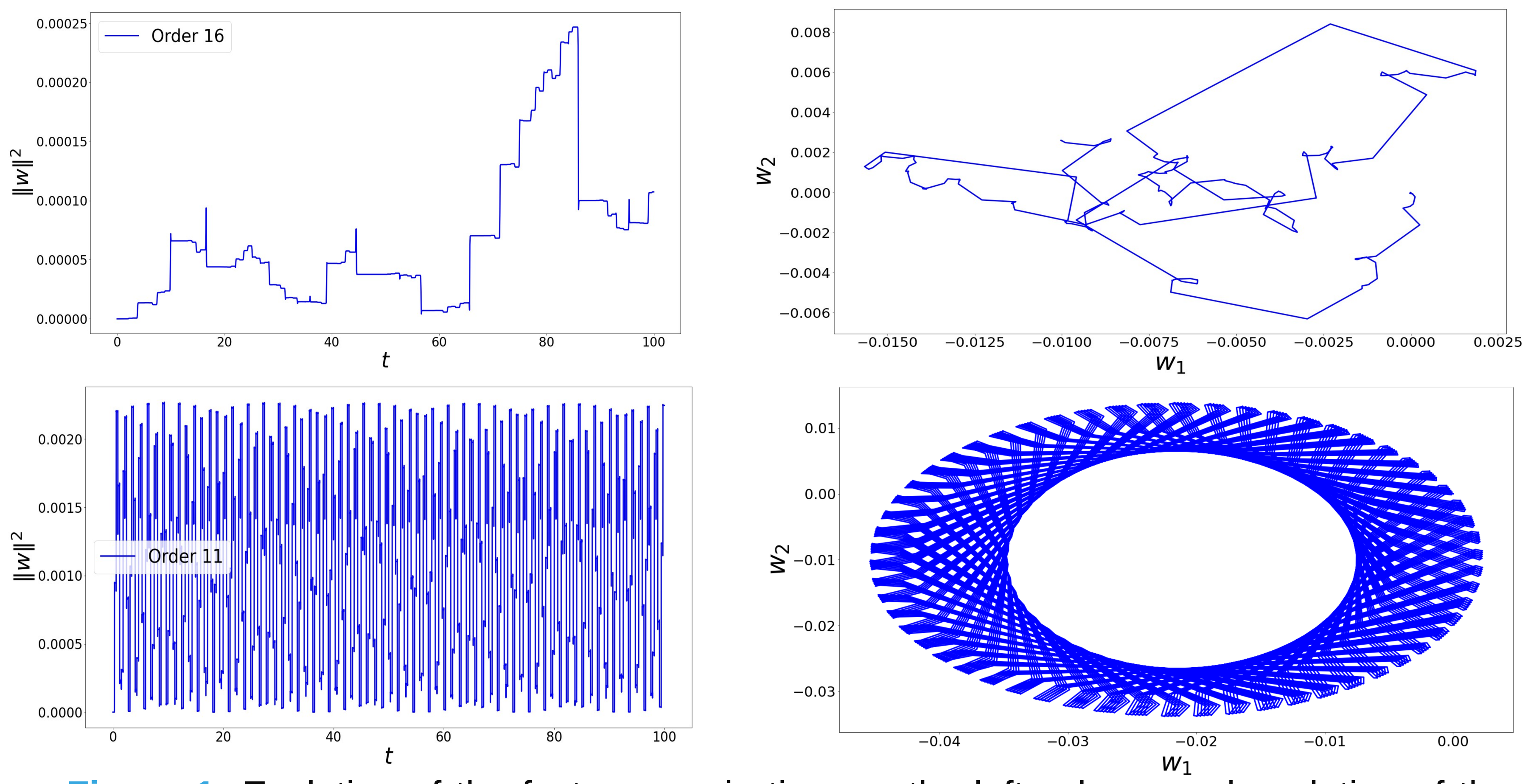

Figure 1: Evolution of the fast energy in time on the left column and evolution of the fast vector field in the phase space on the right column. The first row shows the evolution in a chaotic regime and the second row in a periodic regime.

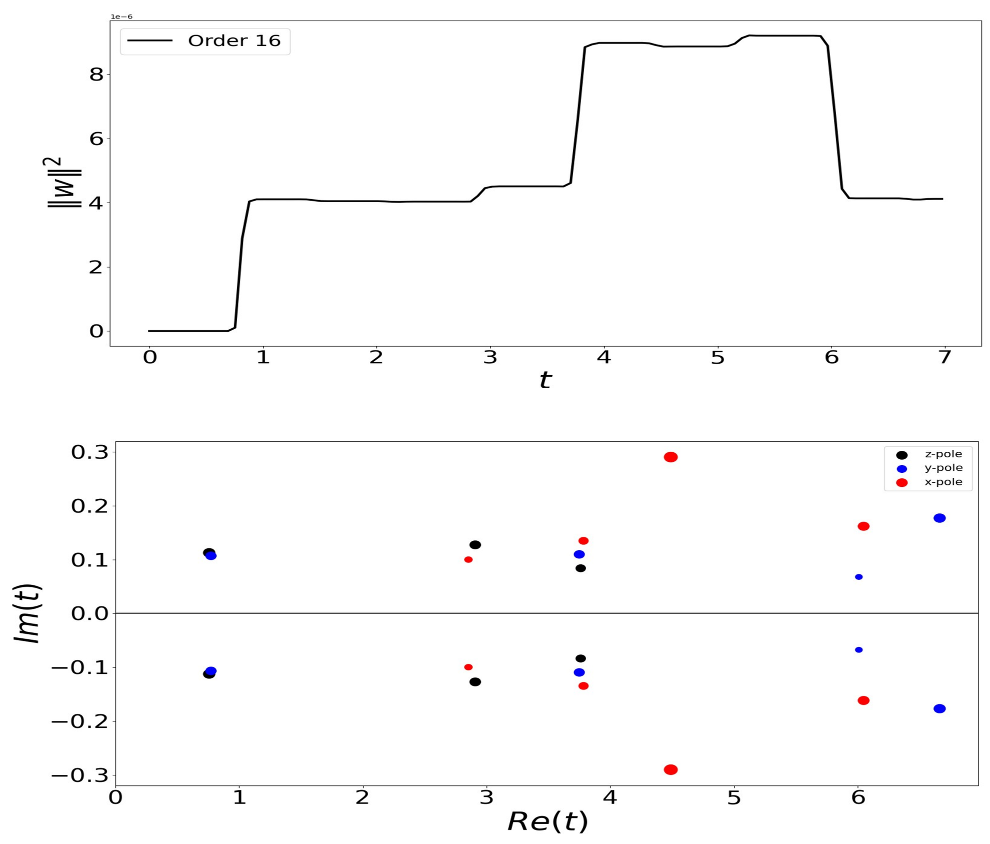

Figure 2: Location of the poles in the complex-t plane using the AAA algorithm for rational interpolation.

Important Observations

- The fast energy evolves in the form of almost discrete jumps with chaotic Lorenz, and this structure is qualitatively different in periodic regimes.
- The fast vector field is well described as a random walk in phase space in chaotic regime.
- Each jump corresponds to a pair of poles of z(t) and the dominant contributions are from the pairs close to the real axis.

Ongoing work

➤ Show by statistic comparison eventually that the slow part of system (The Lorenz'63) coupled with a discrete random walk of step size the amplitude of the fast oscillations computed previously, could surrogate the full system.

➤ Implementation of Optimal Balance on the sphere: Evaluating the change of the picture in Figure 3 in equatorial regions.

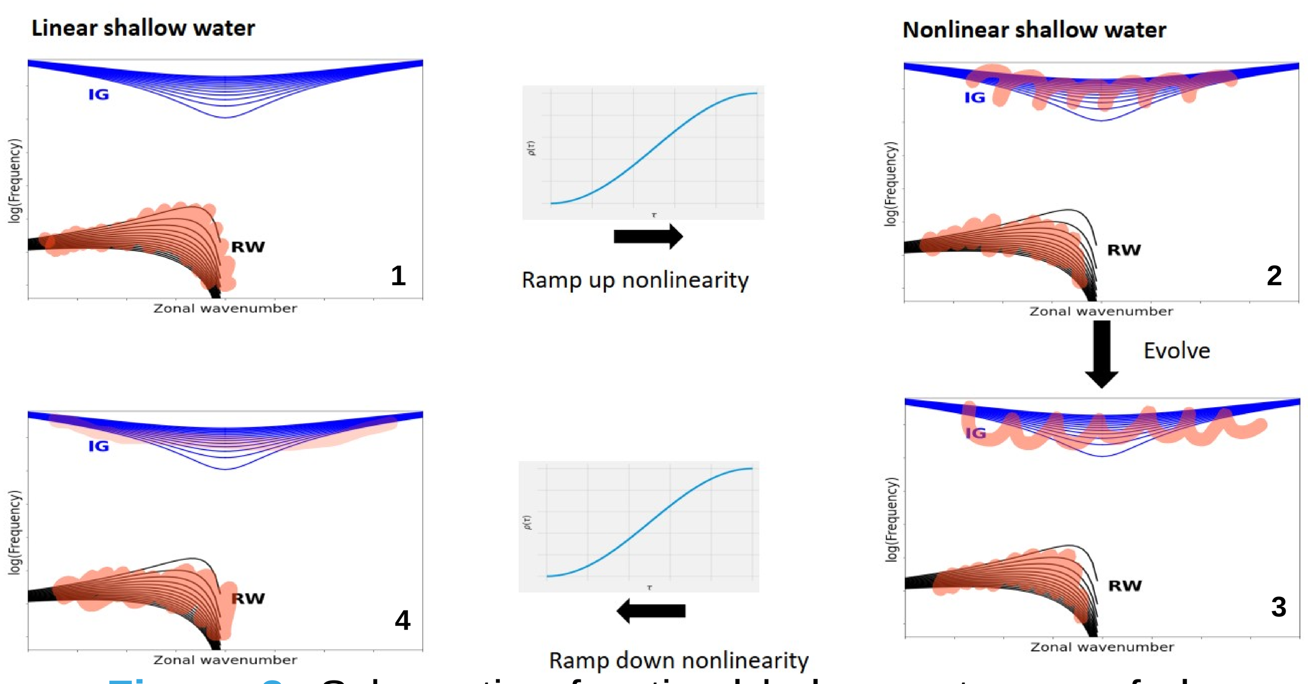

Figure 3: Schematic of optimal balance steps on f-plane shallow water and energy (in red colour) transfer between fast and slow modes during the process.

1. We set up the system in a perfect balanced linear state. All the energy is then in Rossby modes.
2. Slowly ramping up nonlinear interactions excite slaved gravity modes.
3. The system has evolved under full nonlinear dynamics.
4. By slowly ramping down nonlinear interactions, we obtain a state where almost all the energy is back in the Rossby components.

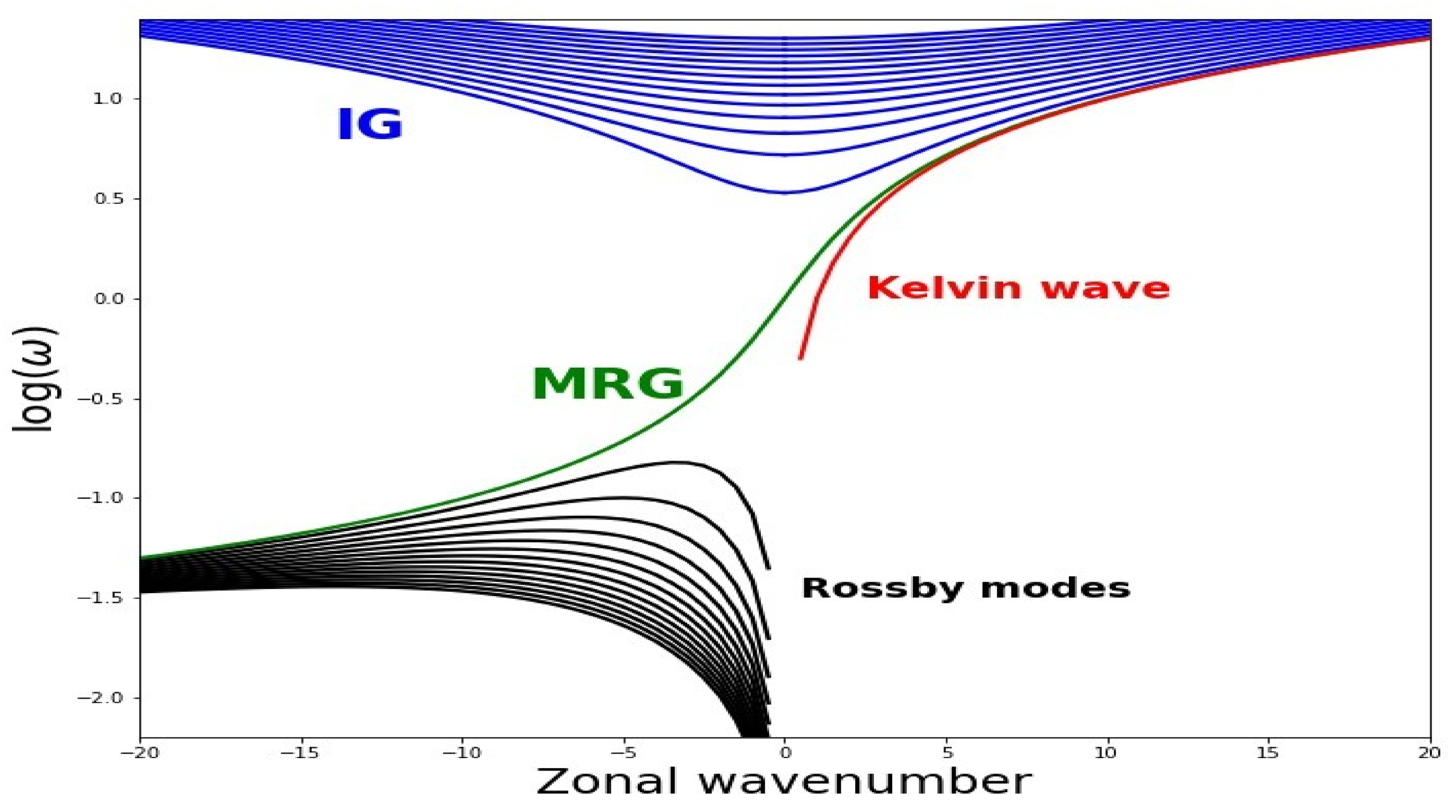

Figure 4: Dispersion relation for the linear waves of the shallow water equations on equatorial regions.