AN ALTERNATIVE APPROACH TO THE OCEAN EDDY PARAMETERIZATION PROBLEM

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Scale interactions, data-driven modeling, and uncertainty in weather and climate

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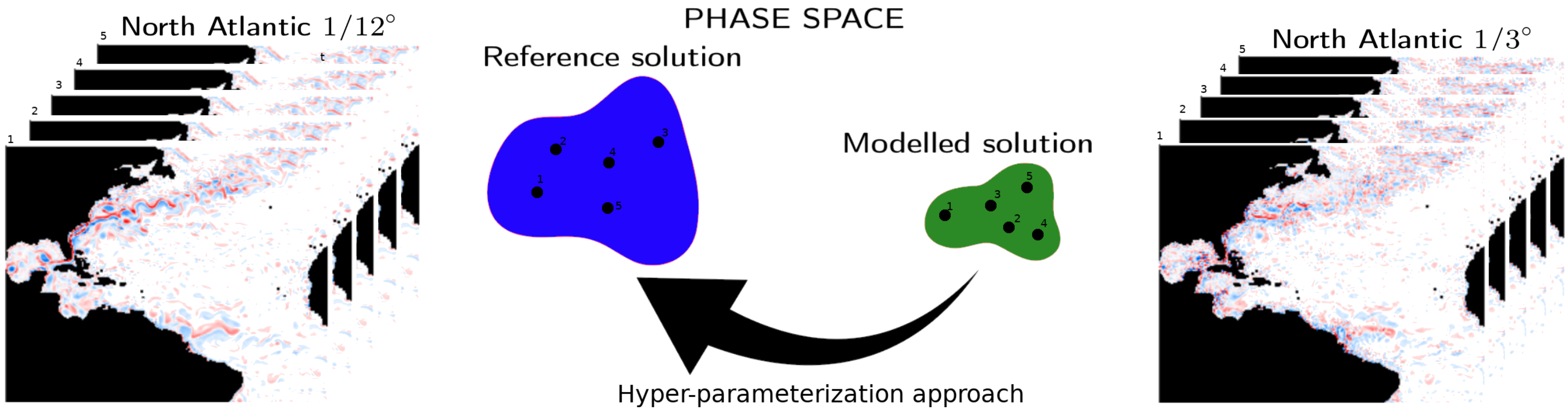
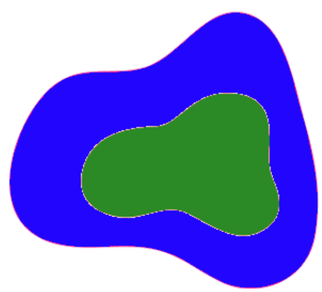
Hyper-Parameterization (HP) approach



Idea: Keep the modelled solution in the region of phase space occupied by the reference solution.



HP draws upon the phase space as an abstraction layer that includes OE effects on all spatiotemporal scales. The main advantage of the HP approach is that it does not require to know the physics behind small scales and large–small scale interactions to reproduce them.

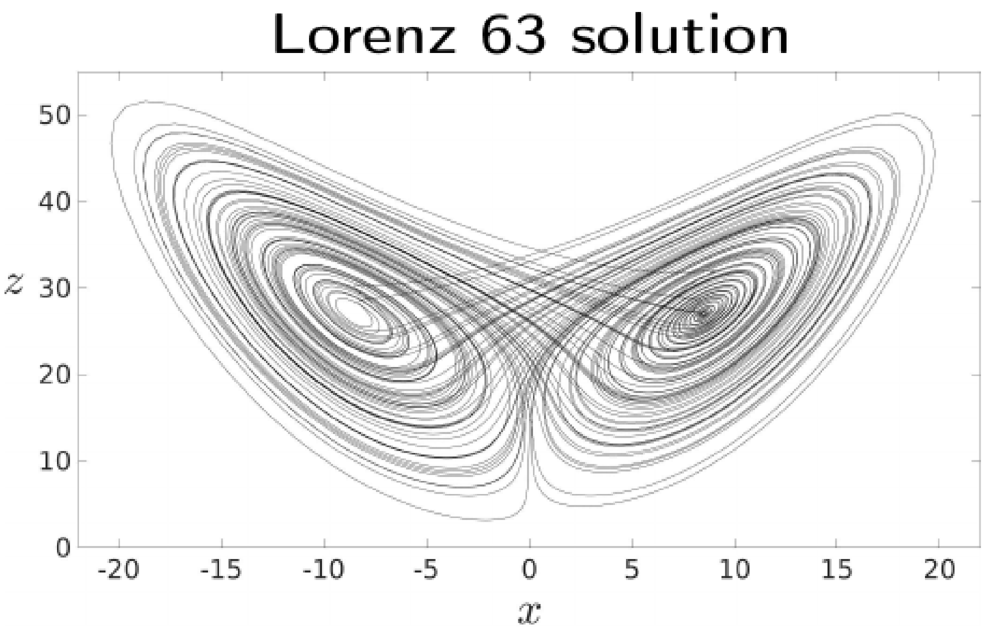
Advection of the image point in phase space

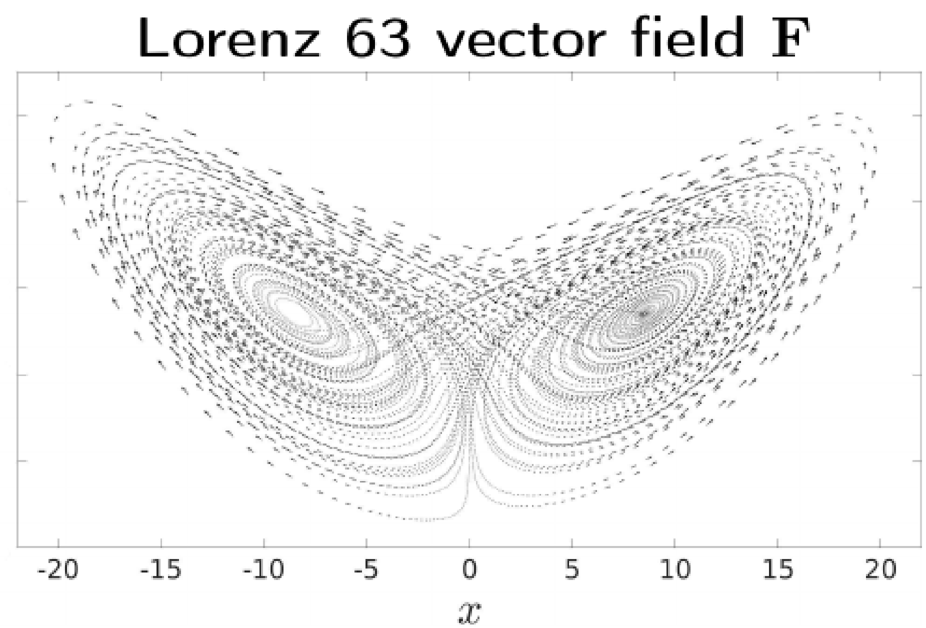
The idea of the method is based on the fact that a first-order ordinary differential equation

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

can be geometrically interpreted as a vector field $\mathbf{F}(\mathbf{x})$ in the phase space. As an example, we consider the Lorenz 63 system:

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)), \quad \mathbf{F} := \begin{pmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{pmatrix}$$



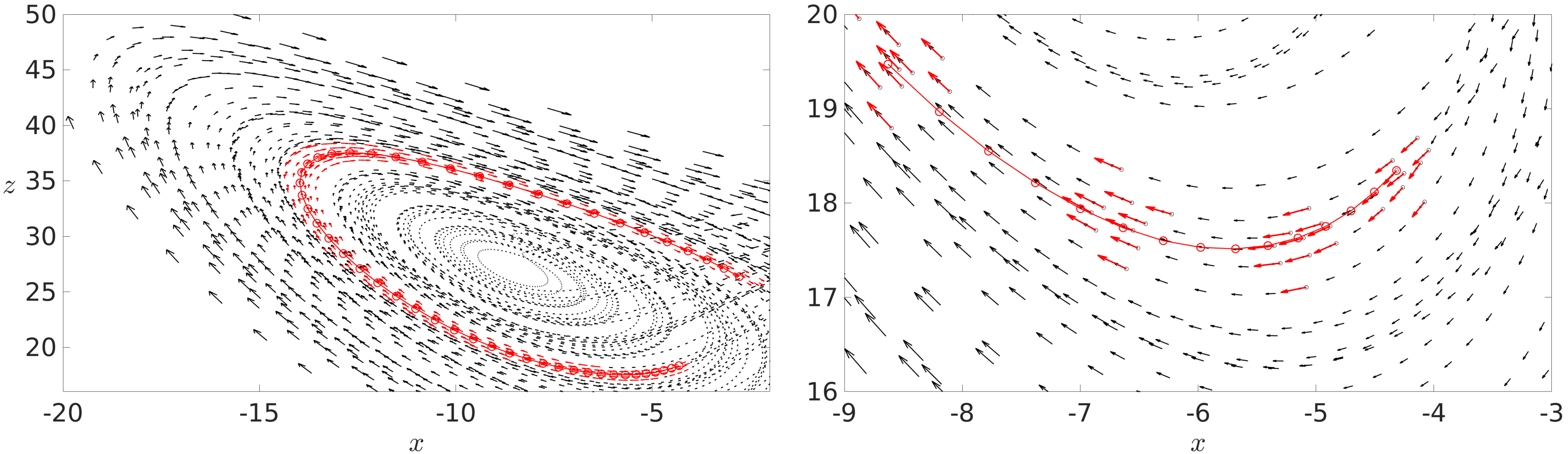


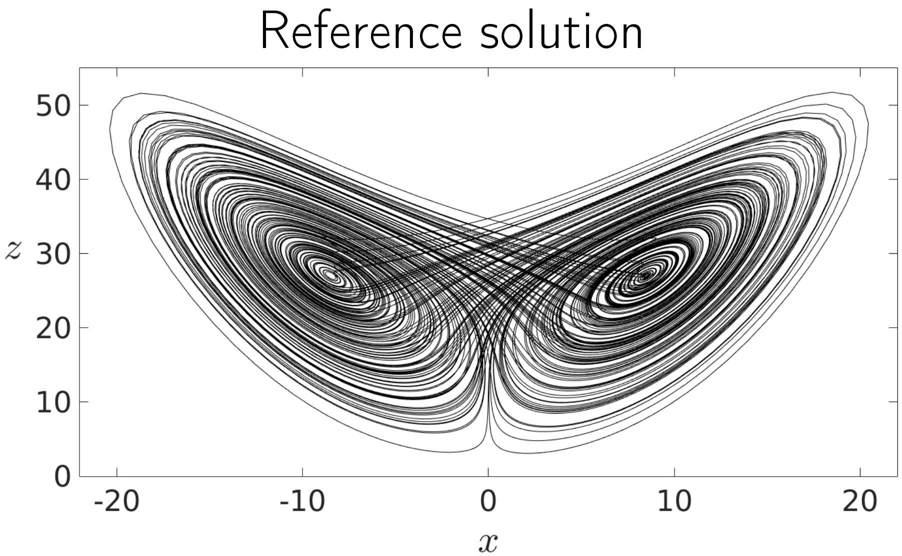
If the vector field $\mathbf{F}(\mathbf{x})$ (computed from the reference data) is known, it can be used it to advect an image point $\mathbf{y}$ (low-resolution solution) the evolution of which can be described by the equation:

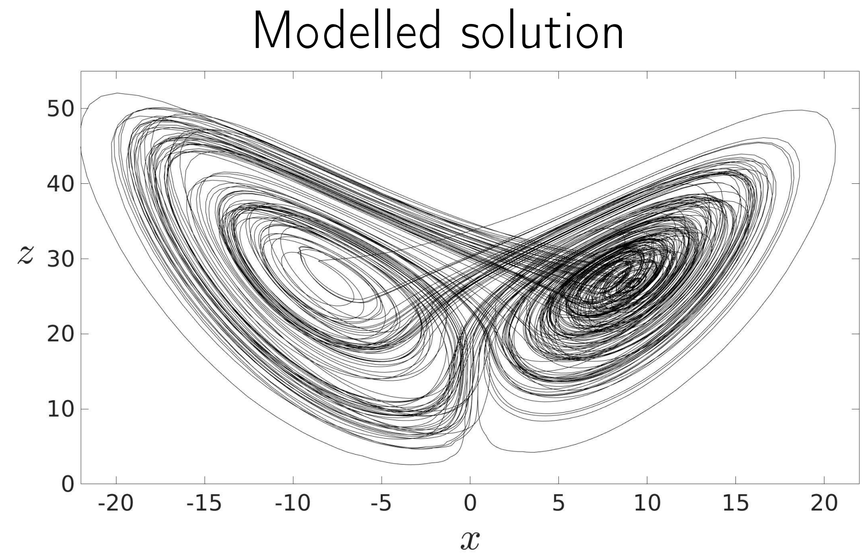
$$\mathbf{y}'(t) = \frac{1}{N} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{F}(\mathbf{x}(t_i)), \quad \mathbf{y}(t_0) = \mathbf{x}(t_0)$$

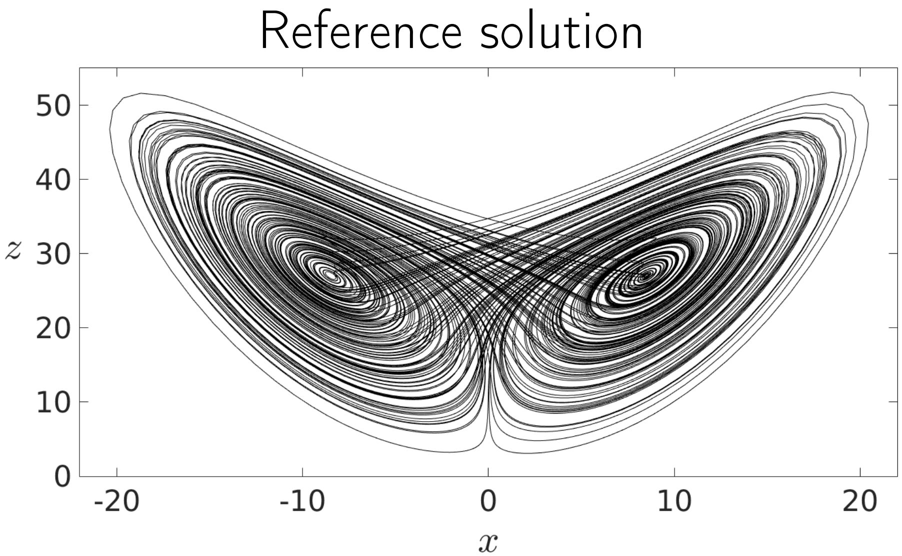
Intuitively, it can be viewed as motion of a ball (image point $\mathbf{y}$) in a river (vector field $\mathbf{F}(\mathbf{x})$), where the nudging term keeps the ball in the river bed.

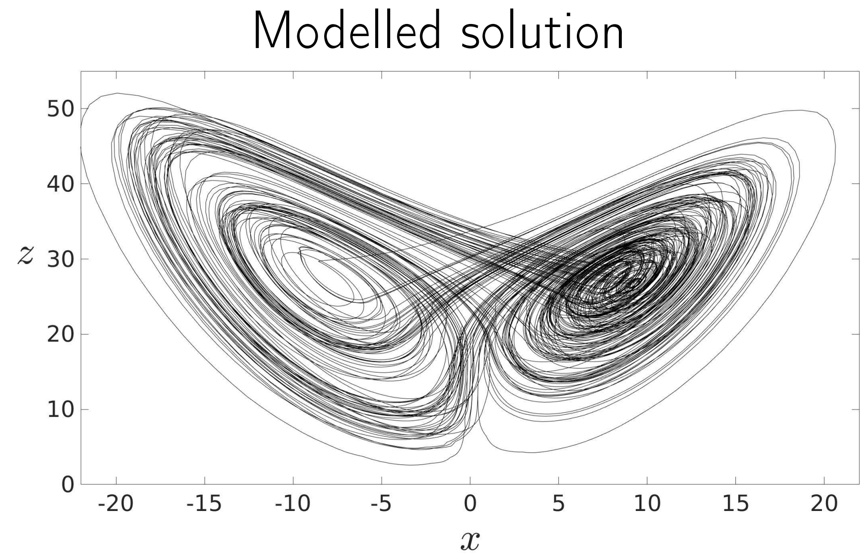
Trajectory $\mathbf{y}(t)$ and its neighbourhood $\mathbf{F}(\mathbf{x}(t_i))\big|_{\mathcal{U}(\mathbf{y}(t))}$ in the Lorenz 63 vector field




Reference solution
z
50
40
30
20
10
0
-20
-10
0
10
20
x

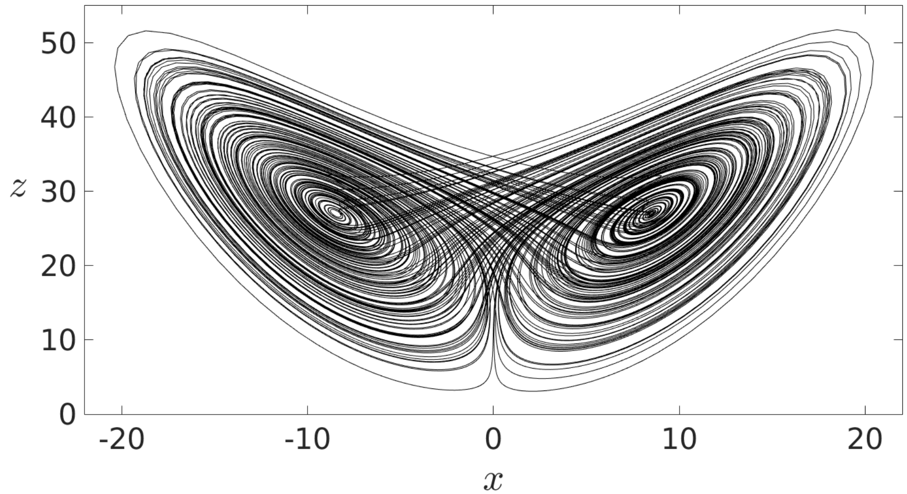

Modelled solution
z
50
40
30
20
10
0
-20
-15
-10
-5
0
5
10
15
20
x



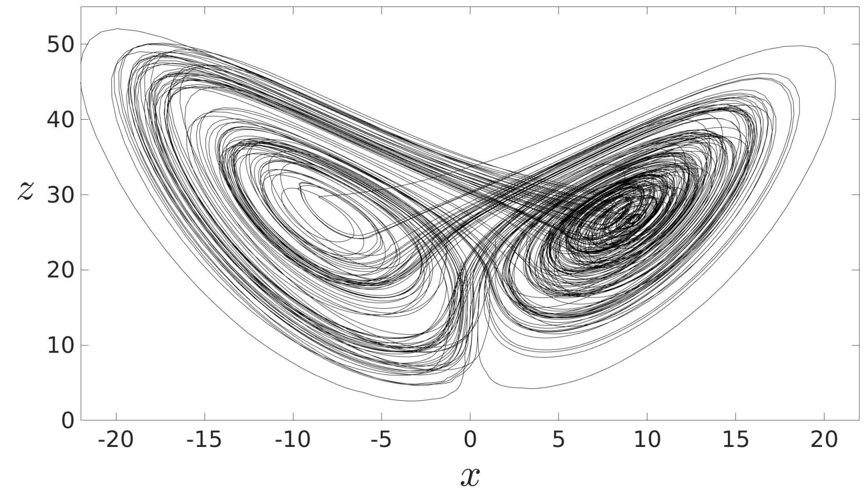


$$\mathbf{y}'(t) = \frac{1}{N_1} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{F}(\mathbf{x}(t_i)) + \eta \left(\frac{1}{N_2} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{x}(t_i) - \mathbf{y}(t) \right), \quad \mathbf{y}(t_0) = \mathbf{x}(t_0)$$

Reference solution

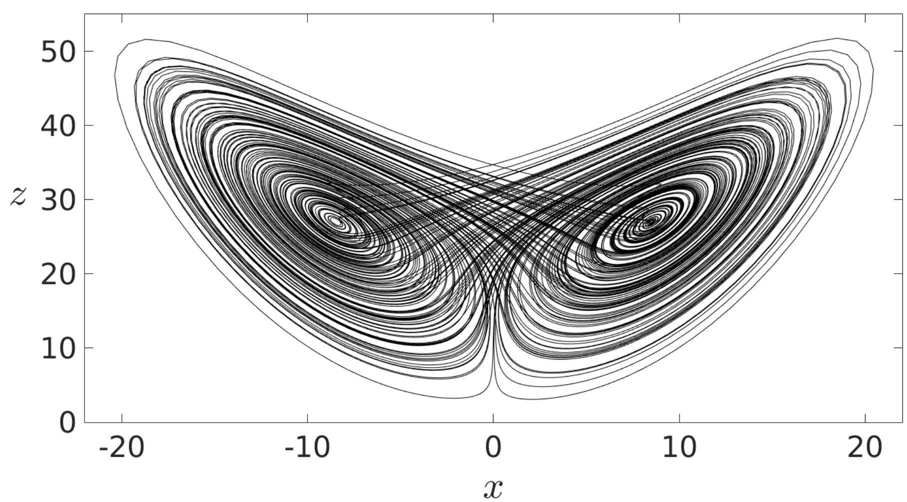


Modelled solution

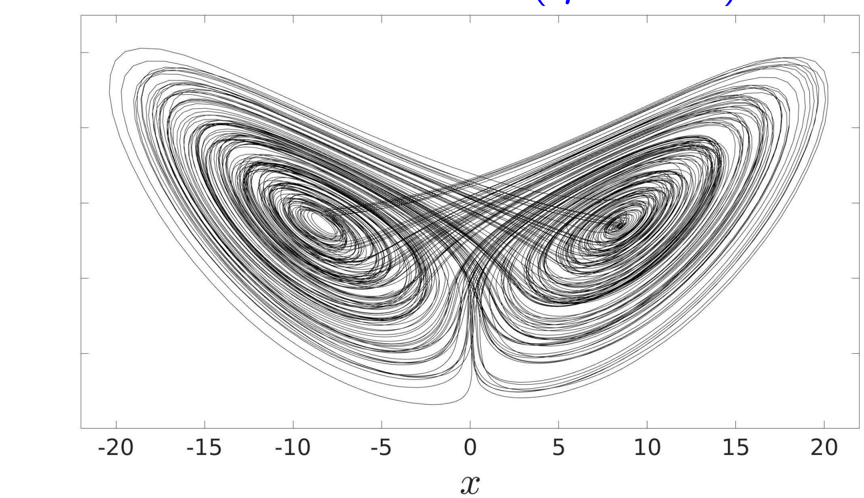


$$\mathbf{y}'(t) = \frac{1}{N_1} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{F}(\mathbf{x}(t_i)) + \eta \left(\frac{1}{N_2} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{x}(t_i) - \mathbf{y}(t) \right), \quad \mathbf{y}(t_0) = \mathbf{x}(t_0)$$

Reference solution



Modelled solution ($\eta = 0.1$)



Multilayer QG equations (channel flow)

$$\begin{aligned} \partial_t q_1 + \mathbf{u}_1 \cdot \nabla q_1 &= \nu \nabla^4 \psi_1 - \beta \partial_x \psi_1, \\ \partial_t q_2 + \mathbf{u}_2 \cdot \nabla q_2 &= \nu \nabla^4 \psi_2 - \mu \nabla^2 \psi_2 - \beta \partial_x \psi_2. \end{aligned} \tag{1}$$

Forcing is given by $\psi_i \to -U_i\, y + \psi_i,\ i = 1, 2$; $\mathbf{q} = (q_1, q_2)$ and $\boldsymbol{\psi} = (\psi_1, \psi_2)$ are related through the system of equations

$$q_1 = \nabla^2 \psi_1 + s_1(\psi_2 - \psi_1), \quad q_2 = \nabla^2 \psi_2 + s_2(\psi_1 - \psi_2). \tag{2}$$

The periodic horizontal boundary conditions set at eastern, Γ_2, and western, Γ_4, boundaries

$$\boldsymbol{\psi}\Big|_{\Gamma_2} = \boldsymbol{\psi}\Big|_{\Gamma_4}, \quad \boldsymbol{\psi} = (\psi_1, \psi_2)\,, \tag{3}$$

and no-slip boundary conditions $$\boldsymbol{u}\Big|_{\Gamma_1} = \boldsymbol{u}\Big|_{\Gamma_3} = 0\,. \tag{4}$$

set at northern, Γ_1, and southern, Γ_3, boundaries of the domain Ω.

The QG equations (18) can be written in the following form

$$\mathbf{q}'(t) = \mathbf{F}(\mathbf{q}, \boldsymbol{\psi}, \mathbf{u}), \tag{5}$$

where $\mathbf{F}$ is the vector field used to advect the image point $\mathbf{y}(t)$ (low-resolution solution):

$$\mathbf{y}'(t) = \frac{1}{N_1} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{F}(\mathbf{q}, \boldsymbol{\psi}, \mathbf{u}) + \eta \left(\frac{1}{N_2} \sum_{i \in \mathcal{U}(\mathbf{y}(t))} \mathbf{q}(t_i) - \mathbf{y}(t) \right), \quad \mathbf{y}(t_0) = \mathbf{q}(t_0).$$

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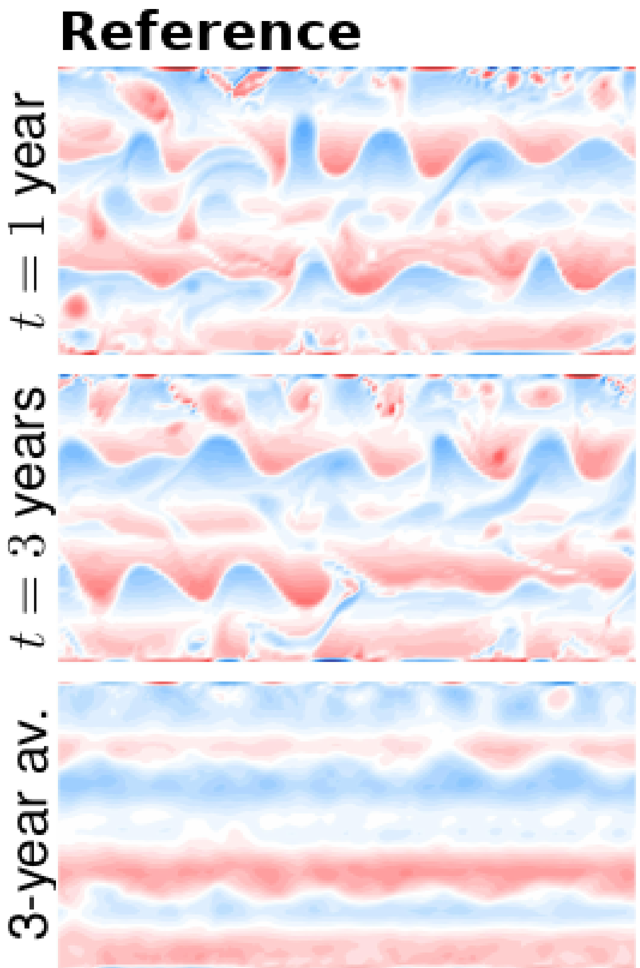
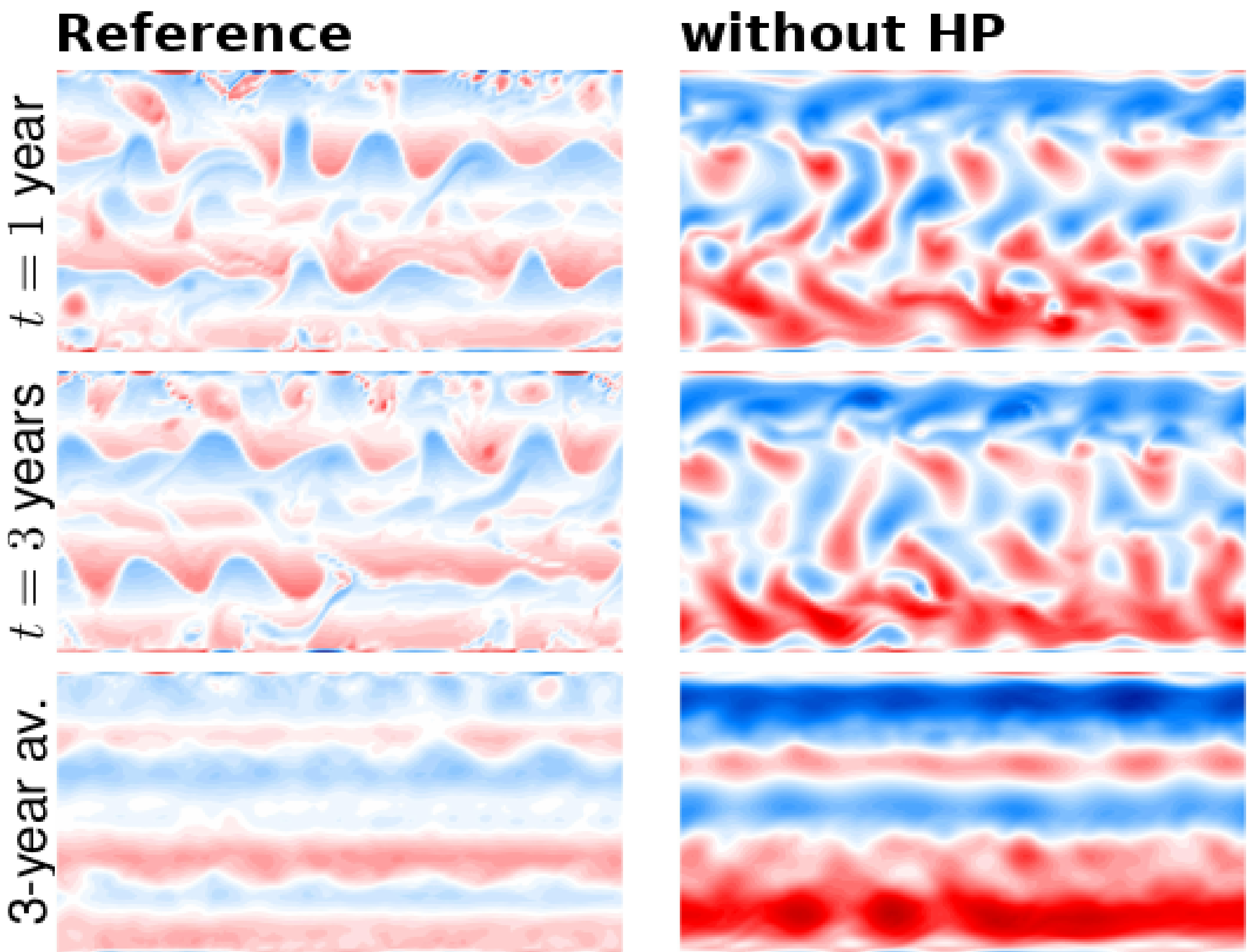


Reference solution: $512 \times 256 \rightarrow 128 \times 64$

Multilayer QG equations (channel flow)



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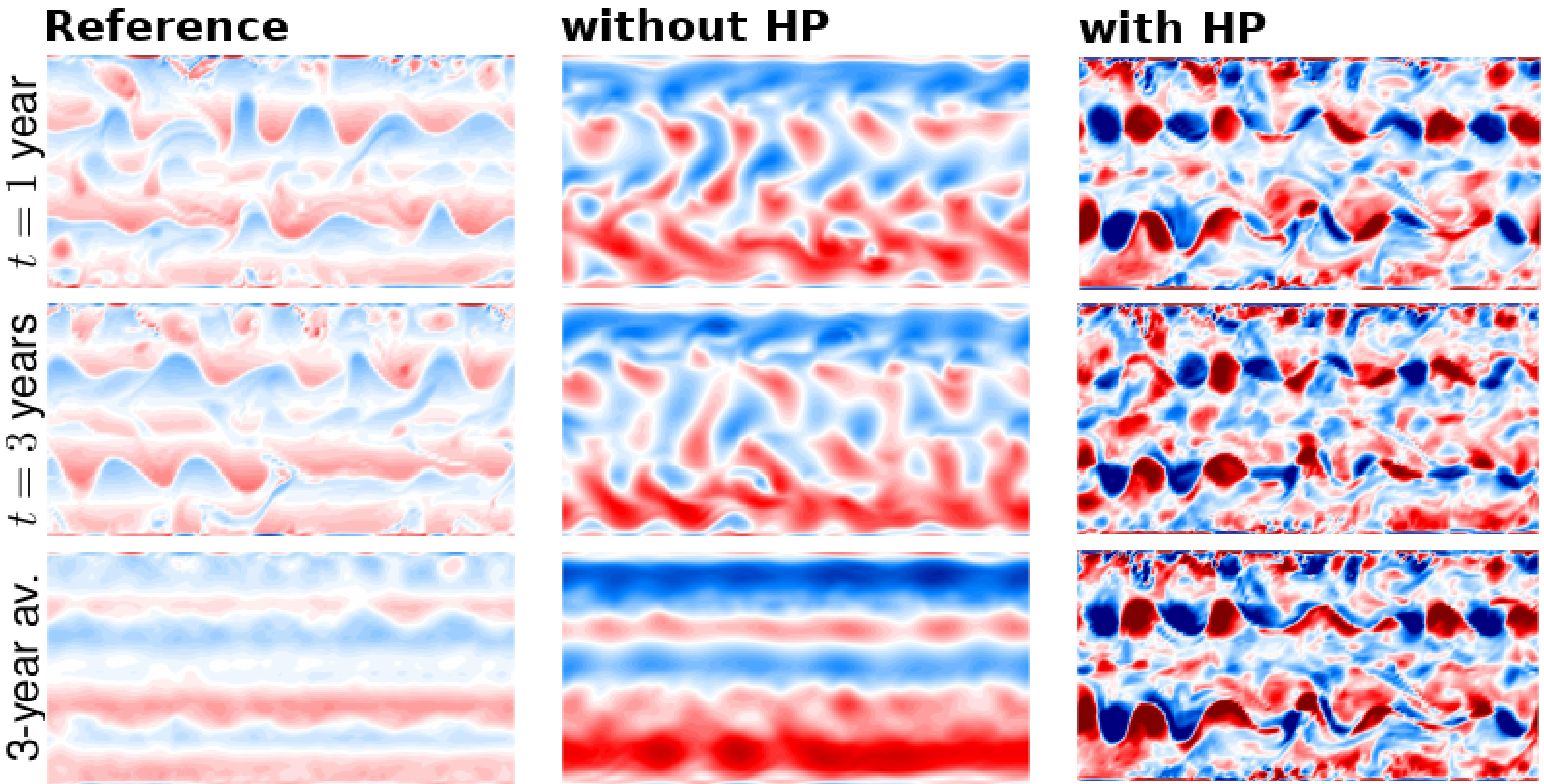
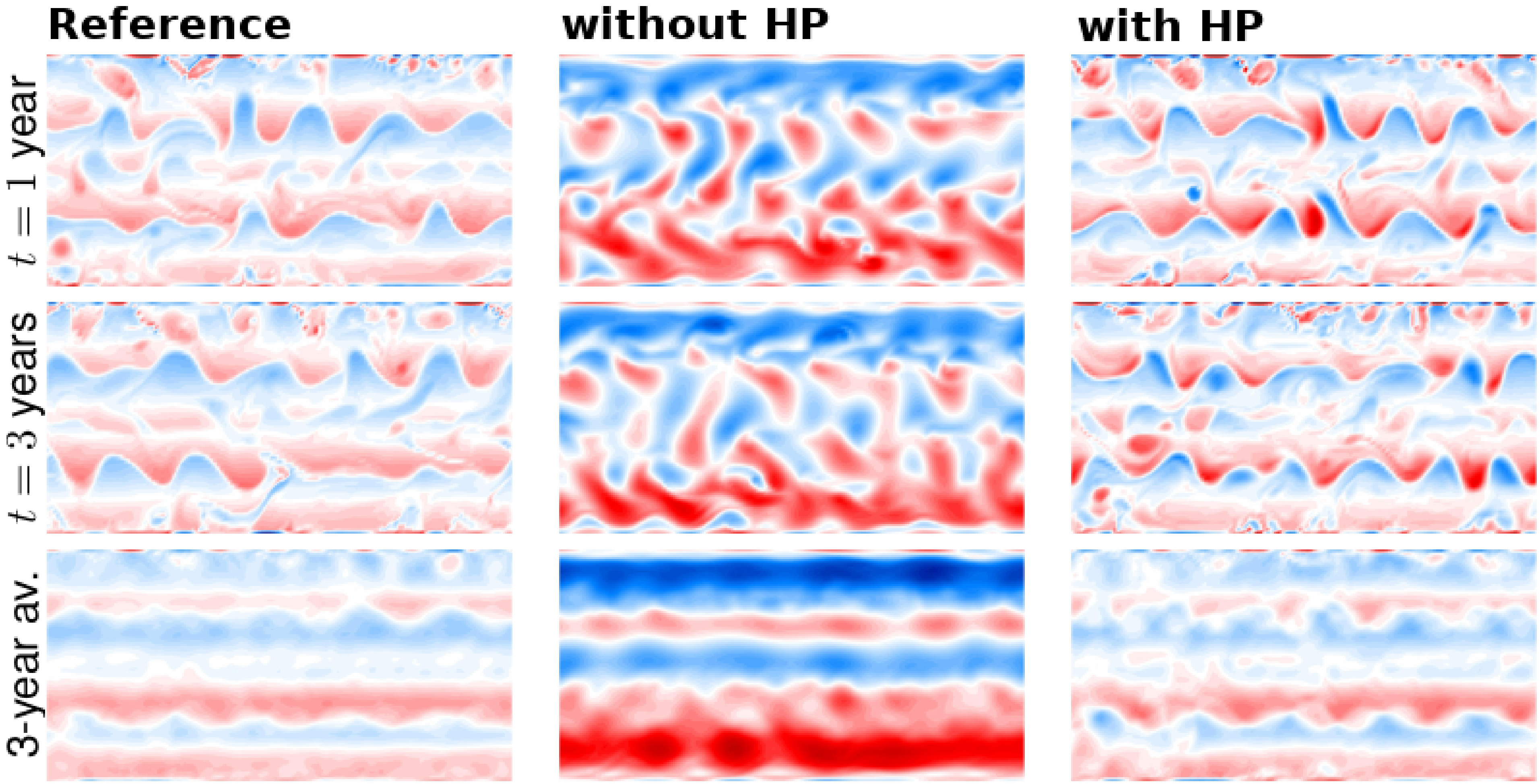
Modelled solution: 128×64

Multilayer QG equations (channel flow)



Reference solution:	$512 \times 256 \to 128 \times 64$
Modelled solution:	128×64
Hyper-parameterized solution ($\eta = 0$):	128×64

Multilayer QG equations (channel flow)



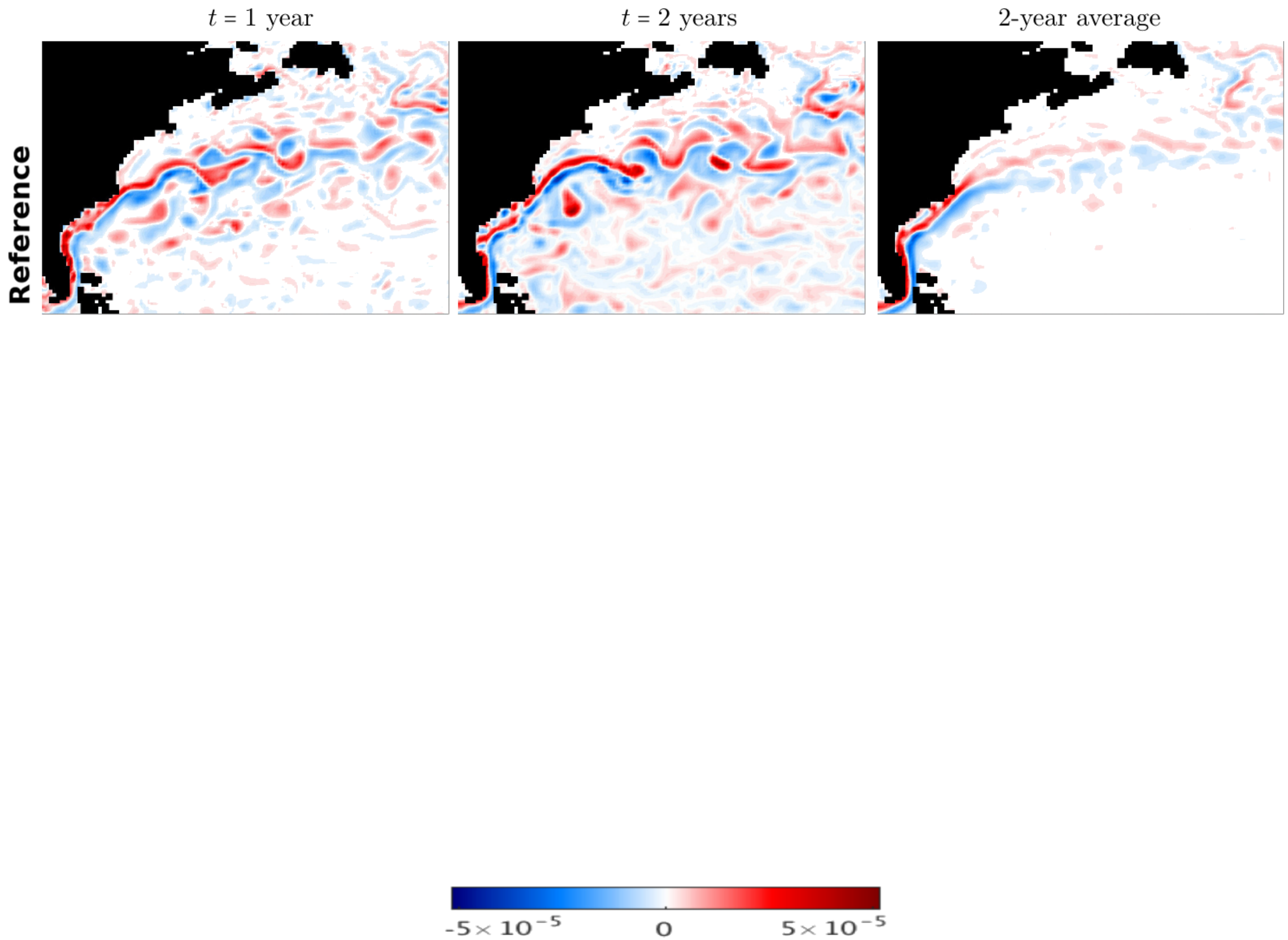
Reference solution: $512 \times 256 \rightarrow 128 \times 64$

Modelled solution: 128×64

Hyper-parameterized solution ($\eta = 0.1$): 128×64

The North Atlantic

A coupled 46-layer ocean-atmospheric model (MITgcm) at $1/12^\circ$ and $1/3^\circ$ horizontal resolution was initially spun up from the state of rest for 5 years, and integrated for another 2 years (1+1 for the hyper-parameterized solution; $\eta = 0.001$).



Reference surface relative vorticity (SRV)

The North Atlantic

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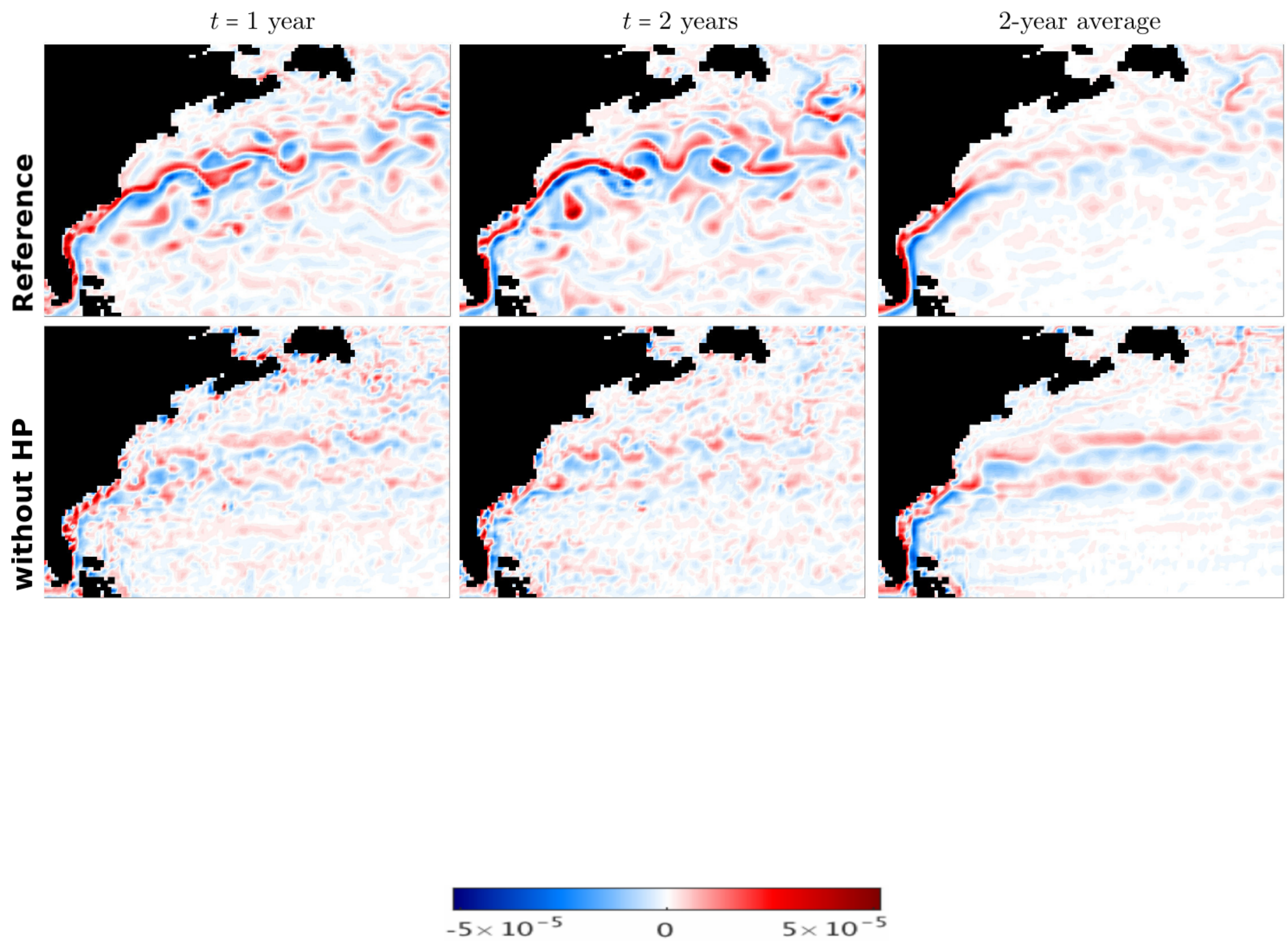

Reference SRV (top), modelled SRV (bottom)

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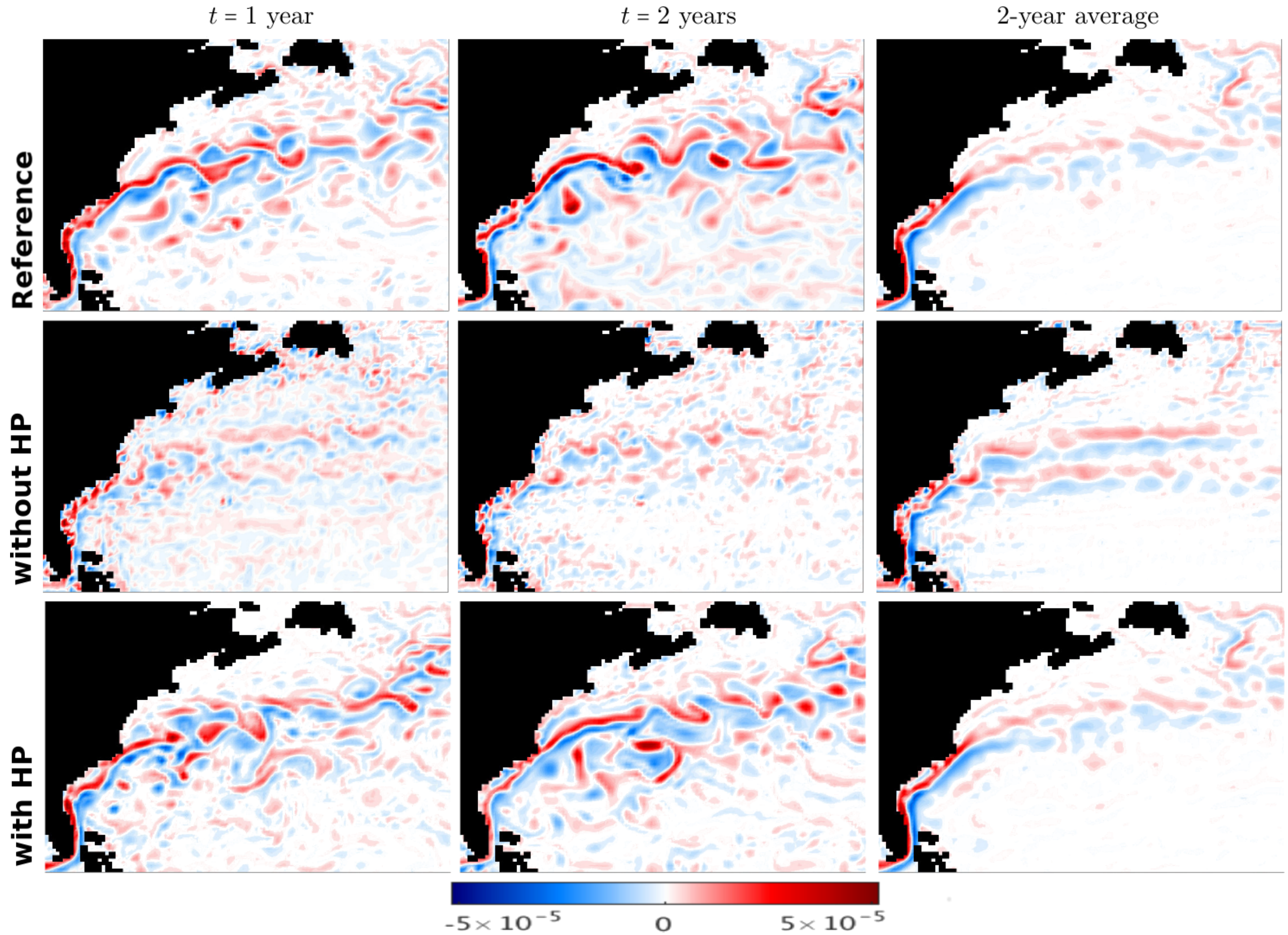
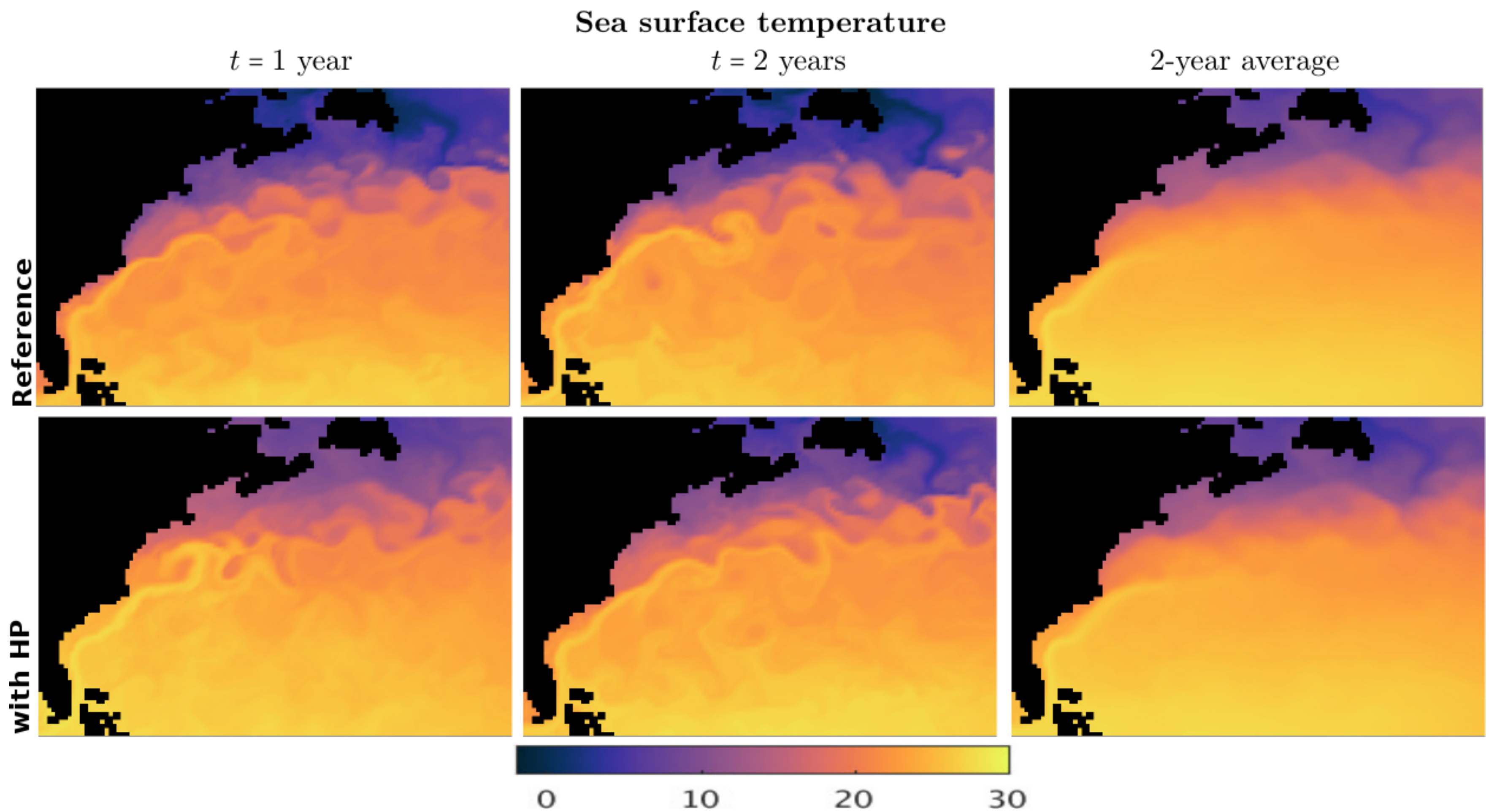
A coupled 46-layer ocean-atmospheric model (MITgcm) at $1/12^\circ$ and $1/3^\circ$ horizontal resolution was initially spun up from the state of rest for 5 years, and integrated for another 2 years (1+1 for the hyper-parameterized solution; $\eta = 0.001$).



Reference SRV (top), modelled SRV (middle), hyper-parameterized SRV (bottom)

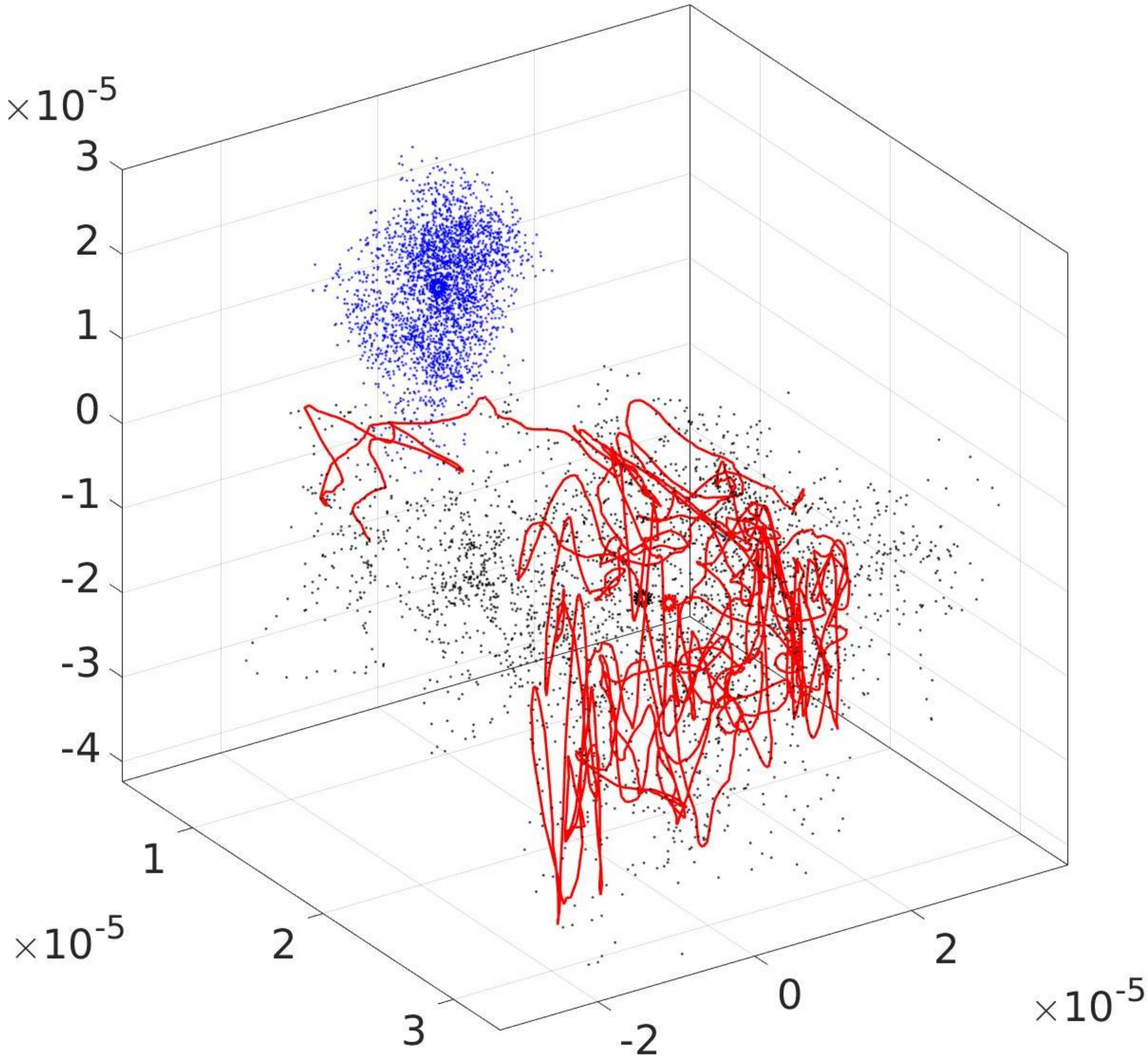
The North Atlantic



Reference SST (top) and hyper-parameterized SST (bottom)

The North Atlantic

Hyper-parameterized solution evolution in the reference phase space



Dynamical system reconstruction

Given a reference solution $\mathbf{x}(t)$, $t \in [0, T]$, $\mathbf{x} \in \mathbb{R}^n$, reconstruct an underlying dynamical system

$$\mathbf{y}'(t) = \mathbf{F}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n, \quad t \in [0, \widetilde{T} > T] \;:\quad \|\mathbf{x}(t) - \mathbf{y}(t)\| \leq \varepsilon$$

Dynamical system reconstruction

Given a reference solution $\mathbf{x}(t)$, $t \in [0, T]$, $\mathbf{x} \in \mathbb{R}^n$, reconstruct an underlying dynamical system (based on a compressed EOF-PC description of $\mathbf{x}$)

$$\mathbf{y}'(t) = \mathbf{F}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^m, \quad t \in [0, \widetilde{T} > T], \quad m << n \ : \quad \|\mathbf{x}(t) - \mathcal{P}\{\mathbf{y}(t)\}\| \leq \varepsilon \,,$$

where $\mathbf{y}(t)$ represents PCs.

Dynamical system reconstruction

Given a reference solution $\mathbf{x}(t)$, $t \in [0, T]$, $\mathbf{x} \in \mathbb{R}^n$, the idea of the method is to reconstruct an underlying dynamical system (based on a compressed EOF-PC description of $\mathbf{x}$)

$$\mathbf{y}'(t) = \mathbf{F}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^m, \quad t \in [0, \widetilde{T} > T], \quad m << n \; : \quad \|\mathbf{x}(t) - \mathcal{P}\{\mathbf{y}(t)\}\| \leq \varepsilon \,, \tag{6}$$

where $\mathbf{y}(t)$ represents PCs. The RHS of (6) is approximated with 2nd order polynomials, $\mathbf{P}(\mathbf{y})$, and Fourier series $\mathcal{F}(\mathbf{y})$:

$$\mathbf{F}(\mathbf{y}) \approx \mathbf{P}(\mathbf{y}) + \mathcal{F}(\mathbf{y}) \,, \tag{7}$$

where

$$\mathbf{P}(\mathbf{y}) := a_0 + \sum_{i=1}^{m} a_i y_i + b_i y_i^2 + c_{ij} y_i y_j, \quad j = 1, \ldots, m, \quad i \neq j, \tag{8}$$

and

$$\mathcal{F}(\mathbf{y}) := \sum_{k=1}^{K} d_k \cos\left(\frac{2\pi k t}{\widetilde{T}}\right) + e_k \sin\left(\frac{2\pi k t}{\widetilde{T}}\right), \tag{9}$$

with unknowns $\mathbf{c} = \{a_0, a_i, b_i, c_{ij}, d_k, e_k\}$, $\quad i, j = 1, \ldots, m (= 30)$, $\quad i \neq j$, and $k = 1, \ldots, K (= 25)$ defined from

$$\mathbf{y}' = \mathbf{A}\mathbf{c}, \tag{10}$$

where $\mathbf{y}'$ is approximated with the forward finite difference over $[0, \widetilde{T}]$ for which the leading PCs (computed from the reference solution) are available.

Having approximated $\mathbf{F}(\mathbf{y})$, we solve the reconstructed dynamical system

$$\mathbf{z}'(t) = \mathbf{P}(\mathbf{z}) + \mathcal{F}(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^m, \quad t \in [0, T], \quad T > \widetilde{T}\,. \tag{11}$$

Once $\mathbf{z}(t)$ is available we compute $\mathbf{x}(t)$ as follows:

$$\mathbf{x}(t) \approx \sum_{i=1}^{m} z_i(t)\mathbf{E}_i\,, \tag{12}$$

with $\mathbf{E}_i$ and z_i being the i-th leading EOF and PC, respectively.

Having approximated $\mathbf{F}(\mathbf{y})$, we solve the reconstructed dynamical system

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with $\mathbf{E}_i$ and z_i being the i-th leading EOF and PC, respectively.

Adaptive nudging. In order to stabilize the numerical integration we use adaptive nudging:

$$\mathbf{z}'(t) = \mathbf{P}(\mathbf{z}) + \mathcal{F}(\mathbf{z}) + \eta(t_i)\left(\frac{1}{N}\sum_{k\in\mathcal{U}(\mathbf{z}(t))}\mathbf{y}(t_k) - \mathbf{z}(t)\right), \quad t \in [0, T]\,, \tag{15}$$

where $\mathcal{U}(\mathbf{z}(t))$ is a neighbourhood of $\mathbf{z}(t)$, and

$$\eta(t_i) = \begin{cases} \eta(t_{i-1}) + \eta_h & \text{if } \sigma(\mathbf{z}(t_i)) > \max_{t\in[0,\widetilde{T}]}\sigma(\mathbf{y}(t)), \\ \eta(t_{i-1}) - \eta_h & \text{if } \sigma(\mathbf{z}(t_i)) \le \max_{t\in[0,\widetilde{T}]}\sigma(\mathbf{y}(t)), \\ 0 & \text{if } \eta(t_{i-1}) - \eta_h < 0. \end{cases} \quad i = 1, 2, \ldots \tag{16}$$

with σ being the standard deviation, $\eta_h = 0.001$, and $\eta(t_0) = 0$.

Multilayer QG equations (idealized Gulf Stream)

A 3-layer QG model for PV anomaly $\mathbf{q} = (q_1, q_2, q_3)$ in Ω:

$$\partial_t q_j + \mathrm{J}(\psi_j, q_j + \beta y) = \delta_{1j} F_{\mathrm{w}} - \delta_{j3}\, \mu \nabla^2 \psi_j + \nu \nabla^4 \psi_j, \quad j = 1, 2, 3\,, \tag{17}$$

where $\mathrm{J}(f,g) = f_x g_y - f_y g_x$, δ_{ij} is the Kronecker symbol, and $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$ is the velocity streamfunction.

$$F_{\mathrm{w}} = \begin{cases} -1.80\, \pi\, \tau_0 \sin\left(\pi y / y_0\right), & y \in [0, y_0), \\ \quad 2.22\, \pi\, \tau_0 \sin\left(\pi (y - y_0)/(L - y_0)\right), & y \in [y_0, L]. \end{cases}$$

$\boldsymbol{q}$ and $\boldsymbol{\psi}$ are coupled through the system of elliptic equations:

$$\boldsymbol{q} = \nabla^2 \boldsymbol{\psi} - \mathbf{S} \boldsymbol{\psi}\,. \tag{18}$$

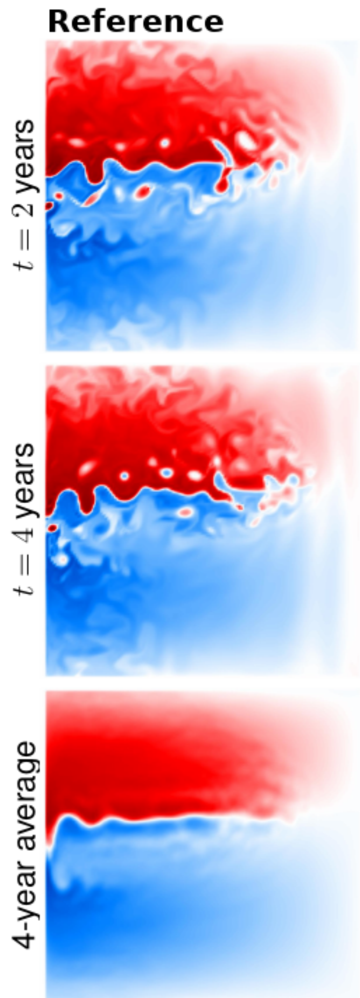
System (17)-(18) is augmented with the integral mass conservation constraint:

$$\partial_t \iint_{\Omega} (\psi_j - \psi_{j+1})\, dy dx = 0, \quad j = 1, 2 \tag{19}$$

with the zero initial condition, and with the partial-slip lateral boundary condition:

$$\left(\partial_{\mathbf{nn}} \boldsymbol{\psi} - \alpha^{-1} \partial_{\mathbf{n}} \boldsymbol{\psi}\right)\Big|_{\partial\Omega} = 0\,. \tag{20}$$

Multilayer QG equations (idealized Gulf Stream)



Reference solution: $512 \times 512 \to 128 \times 128$

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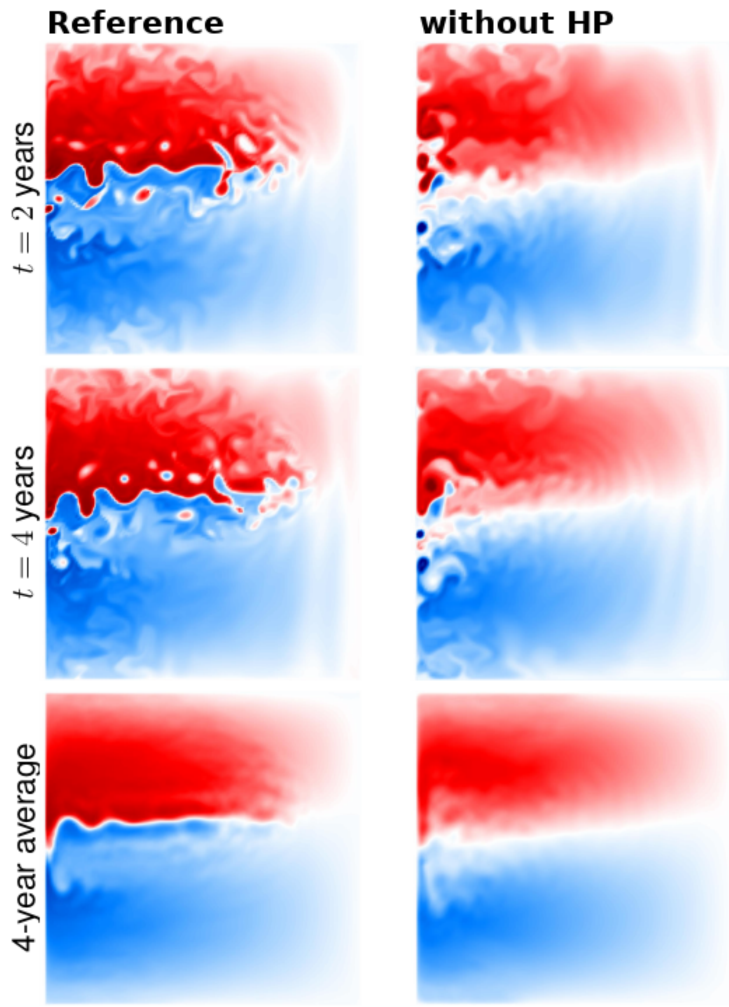
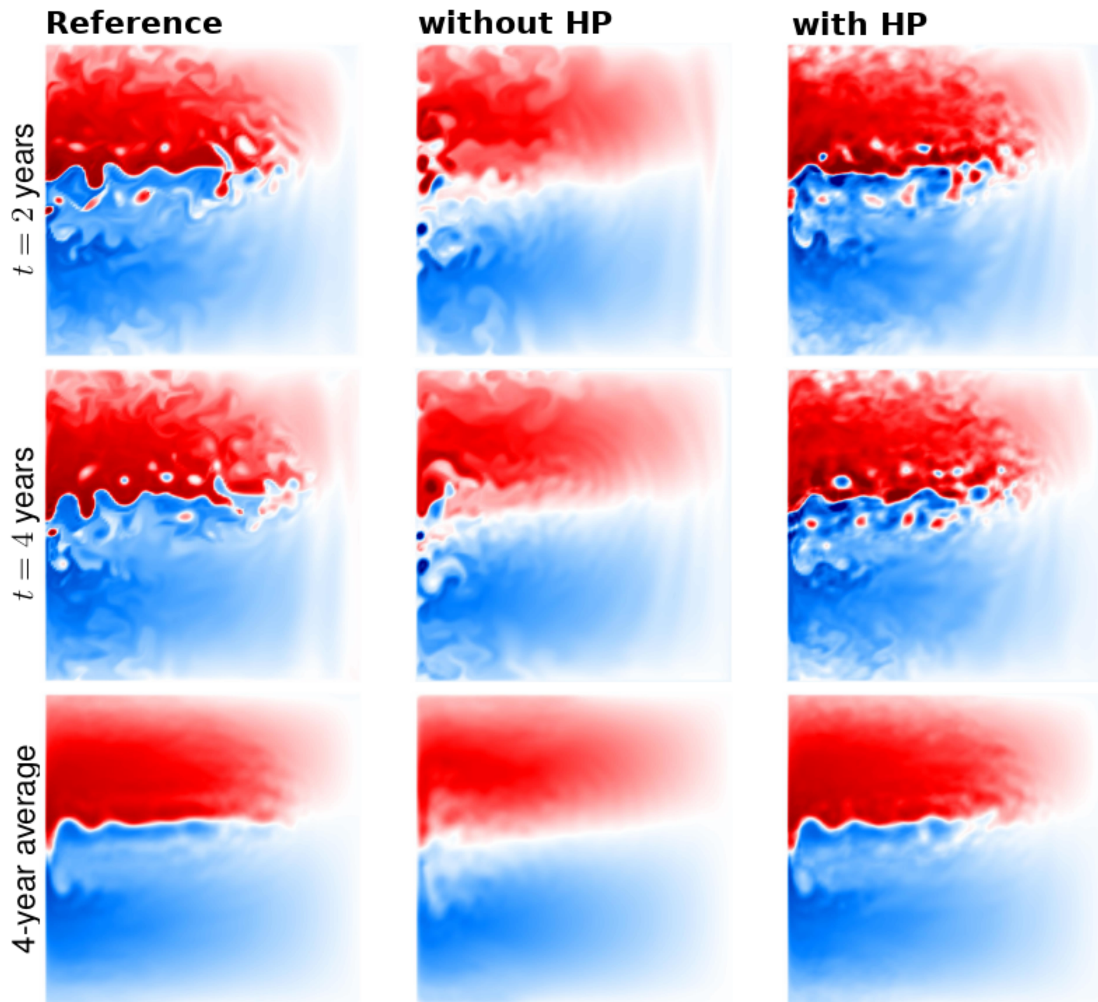


Reference solution: $512 \times 512 \to 128 \times 128$

Modelled solution: 128×128

Multilayer QG equations (idealized Gulf Stream)



Reference solution:	$512 \times 512 \to 128 \times 128$	The dimensionality is reduced by a factor of more than 500
Modelled solution:	128×128	
Hyper-parameterized solution:	128×128	

Constrained dynamics

The idea of the method is to constrain the modelled solution to the reference phase space.

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$$\text{subject to} \quad g(\mathbf{x}) := x^2 + y^2 + z^2 \leq r^2.$$

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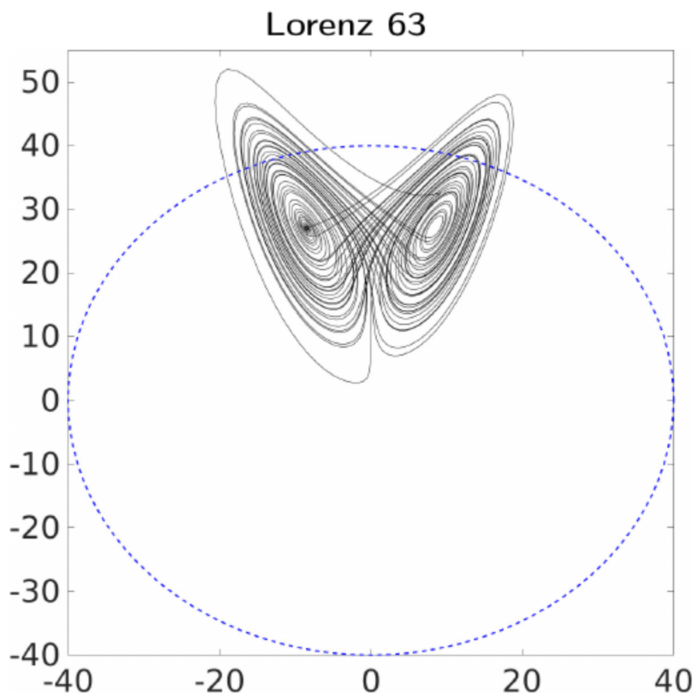

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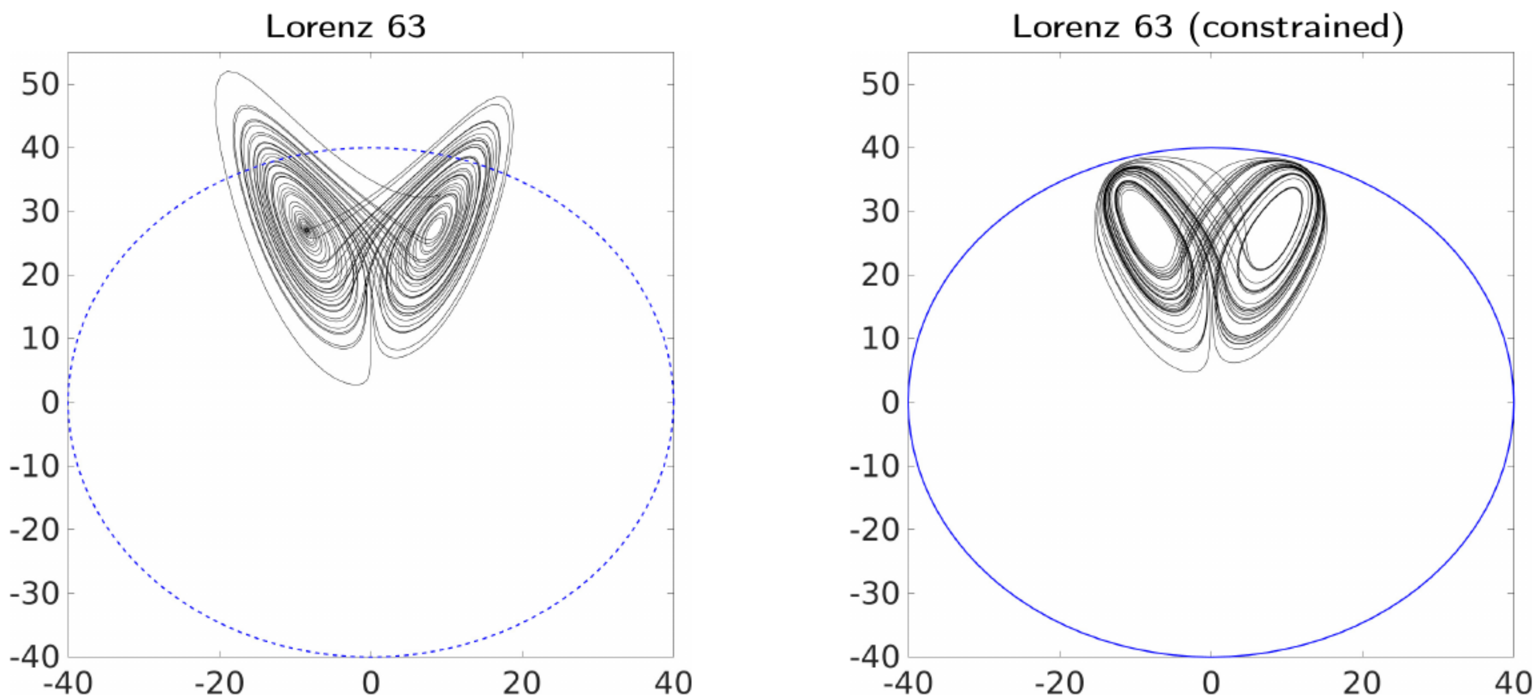
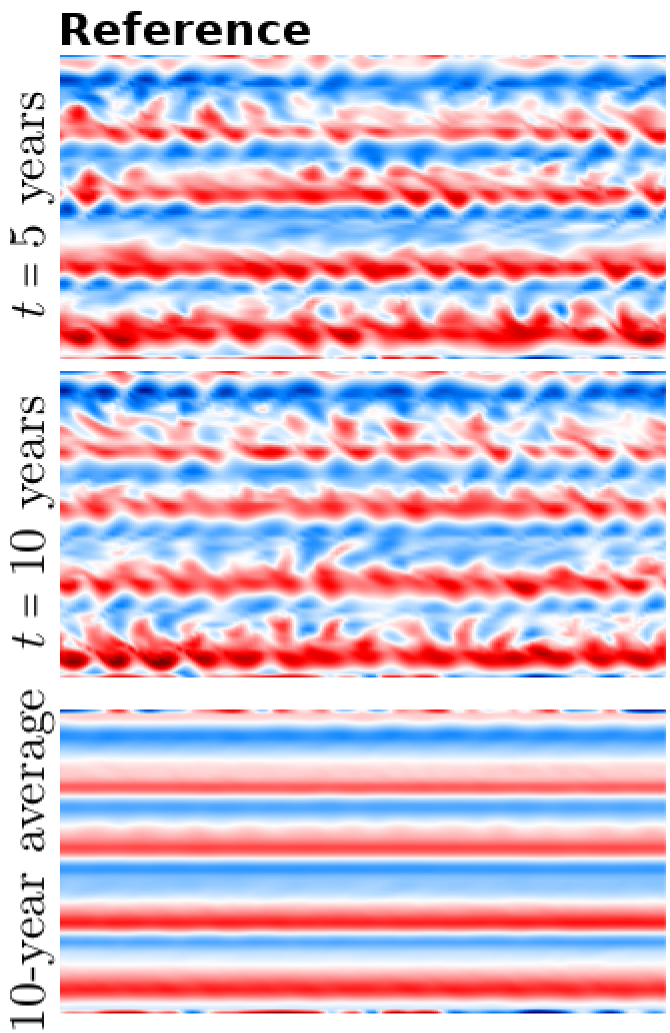
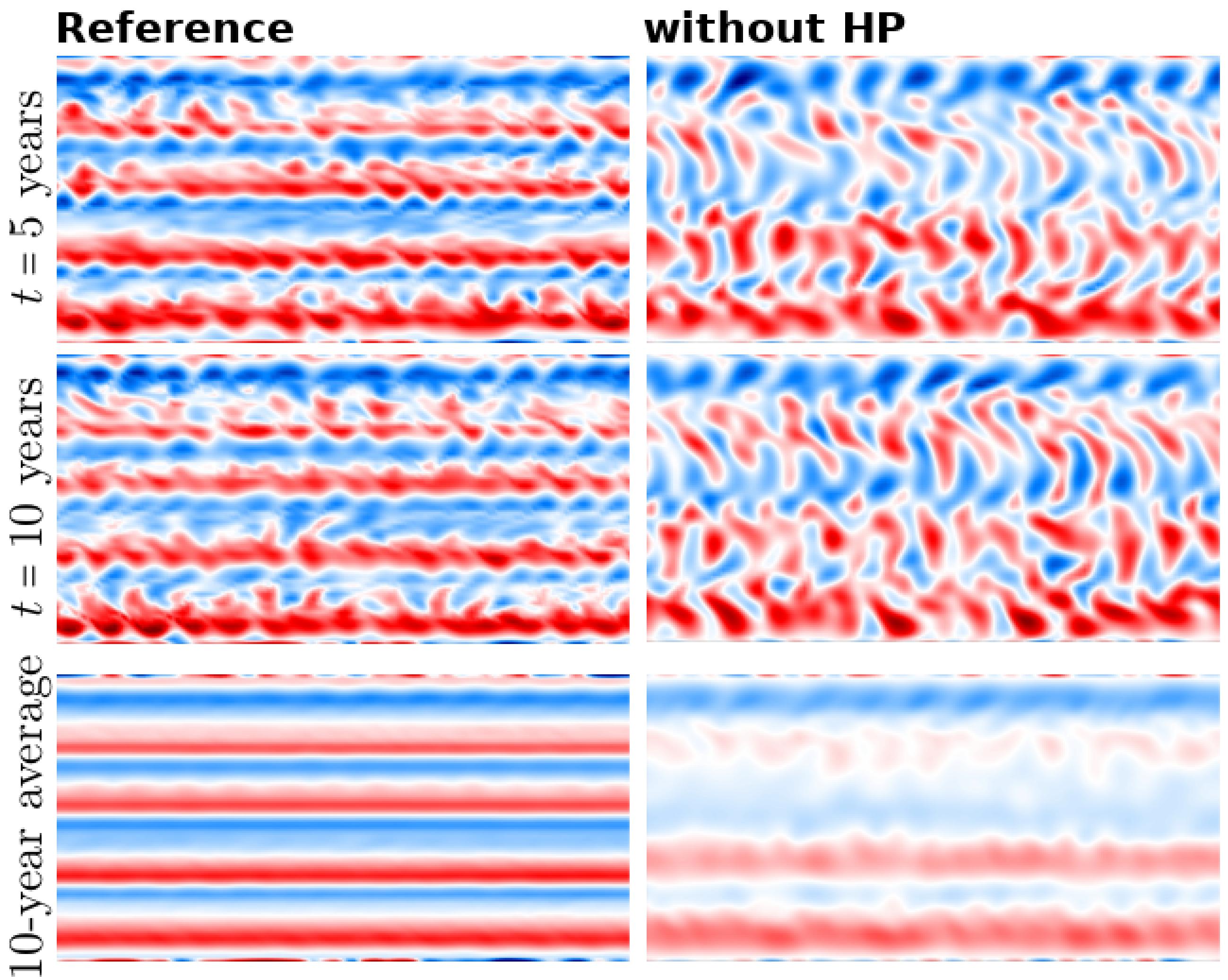


Multilayer QG equations (channel flow)



Reference solution: $512 \times 256 \to 128 \times 64 \quad \nu = 25$

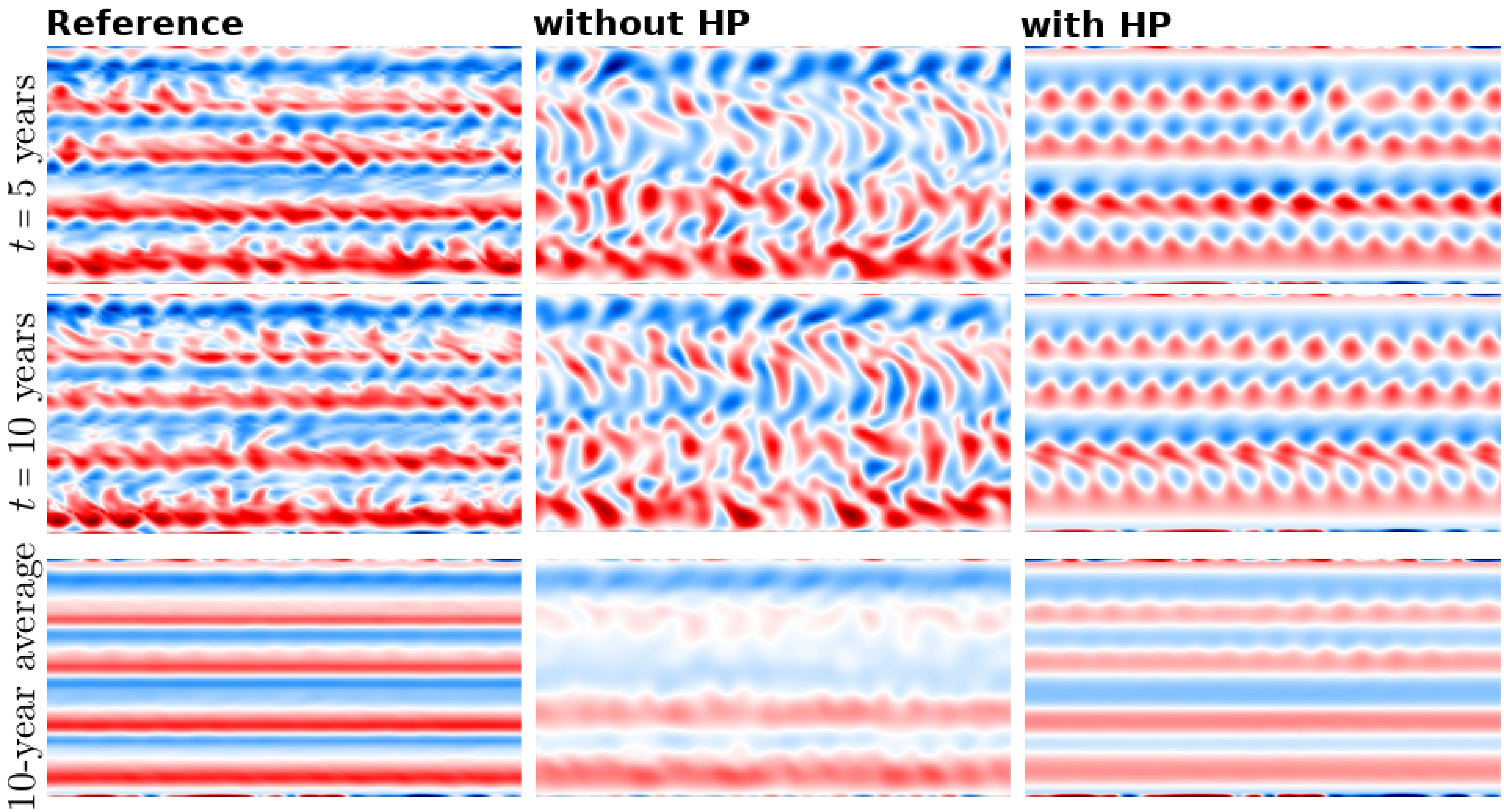
Multilayer QG equations (channel flow)



Reference solution: $512 \times 256 \to 128 \times 64$ $\nu = 25$

Modelled solution: 128×64 $\nu = 250$

Multilayer QG equations (channel flow)



Reference solution:	$512 \times 256 \rightarrow 128 \times 64$	$\nu = 25$
Modelled solution:	128×64	$\nu = 250$
Hyper-parameterized solution:	128×64	$\nu = 250$

ADVANTAGES OF THE HYPER-PARAMETERIZATION APPROACH

- Model choice flexibility (from PDD to HDD)
- Works for both idealized and realistic ocean flows
- Does not require knowledge of physics
- Natural ease of use with comprehensive ocean models
- Can use the reference solution and measurements as input data
- Well-suited for generating ensembles of solutions
- Offers several orders of magnitude acceleration
- Easy to implement