# A Toy Model for Quantification of Unresolved Scales Error Statistics

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## ABSTRACT

In this work, we derive a particle based stochastic model for number concentration and liquid water content using the same assumptions as Wacker and Seifert (2001):

- Extension of definitions to infinitesimally small length scales
- Easily generalizable to more complex and realistic dynamics and hydrometeor geometries
- Explicit distinction between stochastic and deterministic quantities allowing for direct calculation of unresolved scales error covariance matrices
- first DA results considering approximate observation error correlations

## MODEL

We construct a particle based model for **liquid water content L** and **number density N** by replacing the distribution function in the derivation of the model presented in [1] by a (locally finite) counting measure (see e.g. [2] for mathematical background)

$$\mu = \sum_{j=1}^{n} \delta_{(z_j, D_j)},$$

yielding for spherical hydrometeors

$$L(z) = \sum_{j=1}^{n} |P_{x,y}(z) \cap B_{\frac{D_j}{2}}(z_j)|_A$$

and

$$N(z) = \sum_{j=1}^{n} \frac{|P_{x,y}(z) \cap B_{\frac{D_j}{2}}(z_j)|_A}{|B_{\frac{D_j}{2}}(z_j)|_V},$$

with $z$ denoting height, $B_R(x)$ a ball of radius $R$ around $x$, $P_{x,y}(z)$ the $(x,y)$-plane at height $z$ and $|.|_A$ and $|.|_V$ area and volume. This, assuming constant movement of hydrometeors with their terminal fall velocity $v_T$, yields the dynamics

$$z_j(t) = z_j^0 + v_T(D_j)t,$$

which allows to rewrite $L(z,t)$ and $N(z,t)$ by expressions of the form

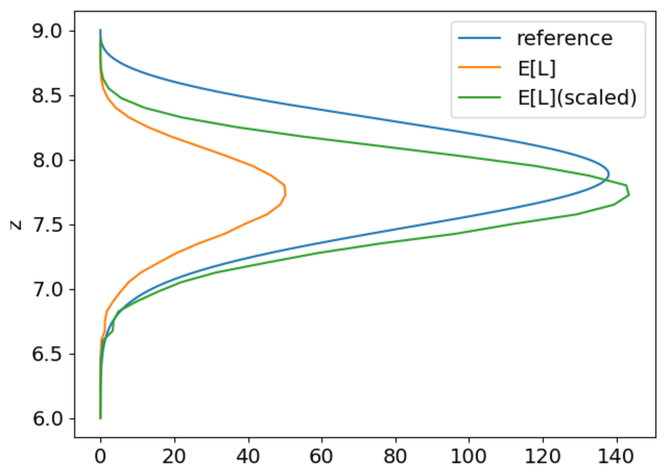
$$Z_{z,t} = \sum_{j=1}^{n} I_{\left[z-\frac{D_j}{2}, z+\frac{D_j}{2}\right]}(z_j(t)) g_Z(D_j) f(D_j, z_j, t, z),$$

with $g_Z$ and $f$ functions depending on observable and particle geometry. Given stochastic initial conditions this allows for the calculation of the expectation value $\mathbb{E}[Z_{z,t}]$ and covariances $Cov(X_{x,t}, Y_{y,s})$.

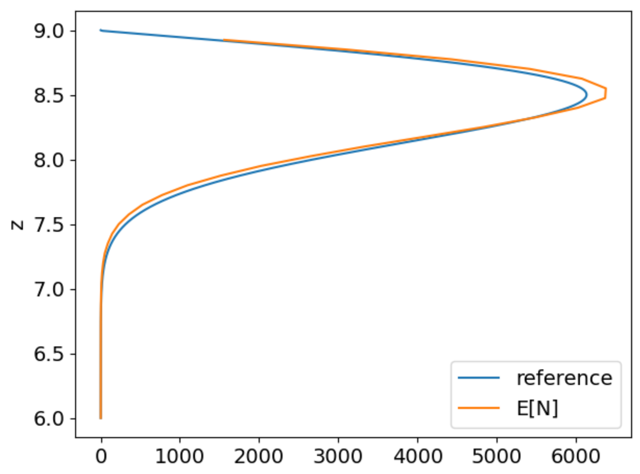
Replacing the Ball in the definitions of L and N by other object allows to treat arbitrary geometries.

## LIQUID WATER CONTENT AND NUMBER DENSITY PROFILES

We calculate the liquid water content and number density profiles for different particle geometries assuming volume equal to spheres with diameters distributed $\sim e^{-\lambda D}$:

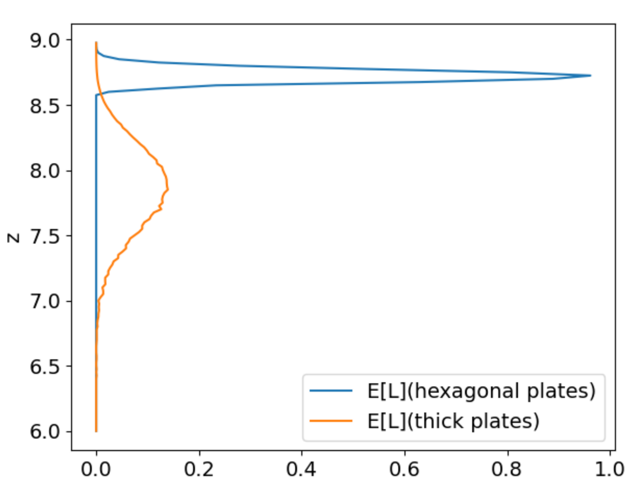



Liquid water content for spherical hydrometeors with $\lambda = 70\frac{1}{m}$

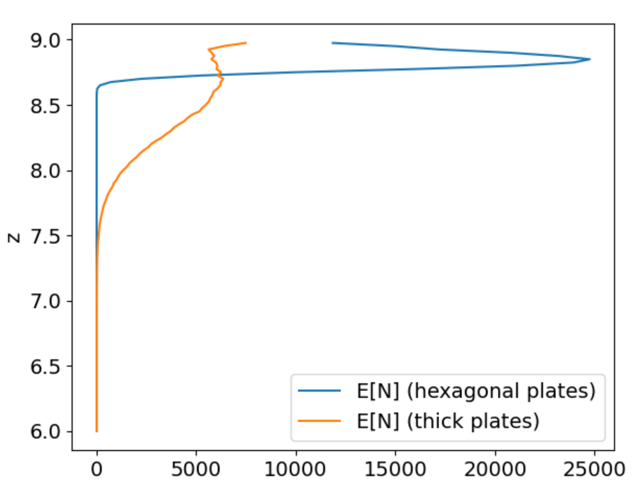



Number density for spherical hydrometeors with $\lambda = 70\frac{1}{m}$

To calculate the profiles for cylindrical hydrometeors, we assume the axis ratios to be following the power laws given in [3] and the terminal fall velocities by the equations given in [4] :

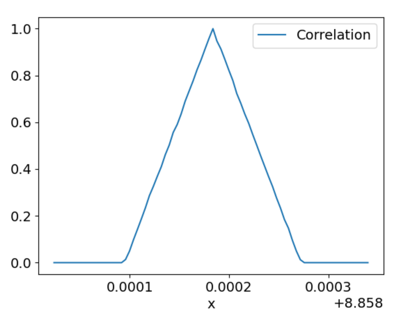



Liquid water content for cylindrical hydrometeors with $\lambda = 500\frac{1}{m}$

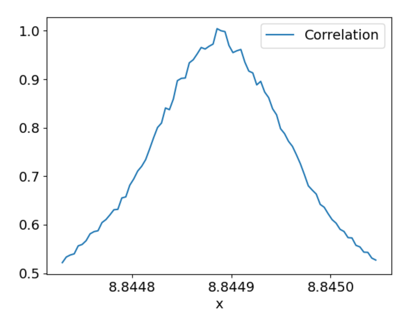



Number density content for cylindrical hydrometeors with $\lambda = 500\frac{1}{m}$

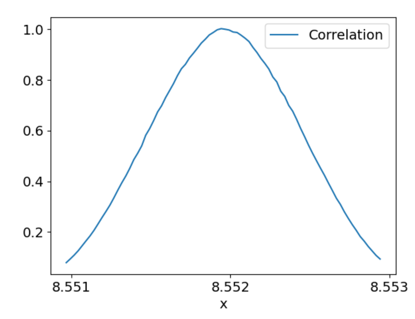
## ERROR COVARIANCE MATRICES




Correlation profile for cylindrical hydrometeors with $\phi = 90°$

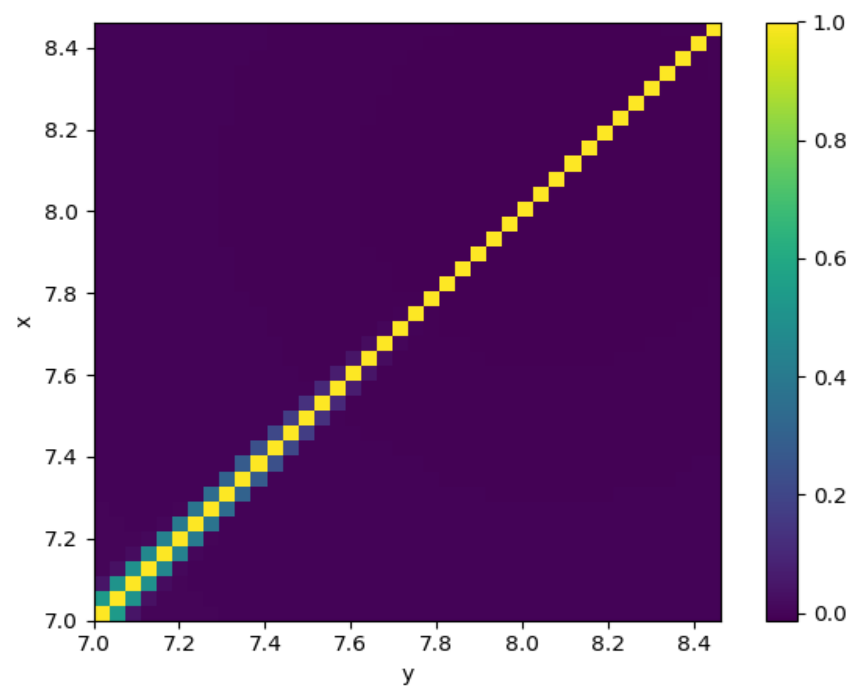



Correlation profile for cylindrical hydrometeors with $\phi \sim U([0°, 90°])$




Correlation profile for spherical hydrometeors

We can calculate the error covariance matrices associated with liquid water content and number density profiles for different particle geometries.

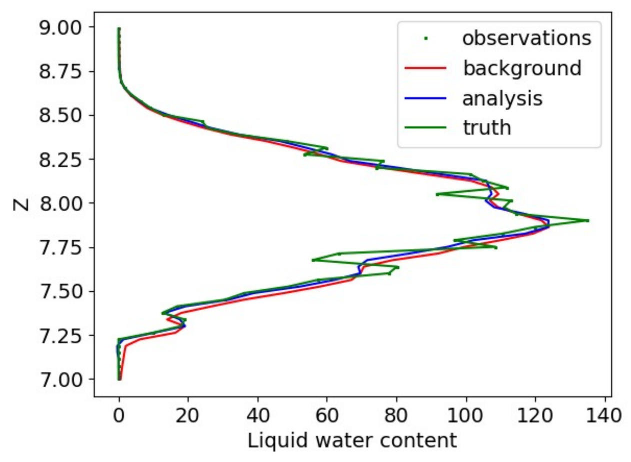



Large scale structure of the (N,N)-part of the correlation matrix for spherical hydrometeors

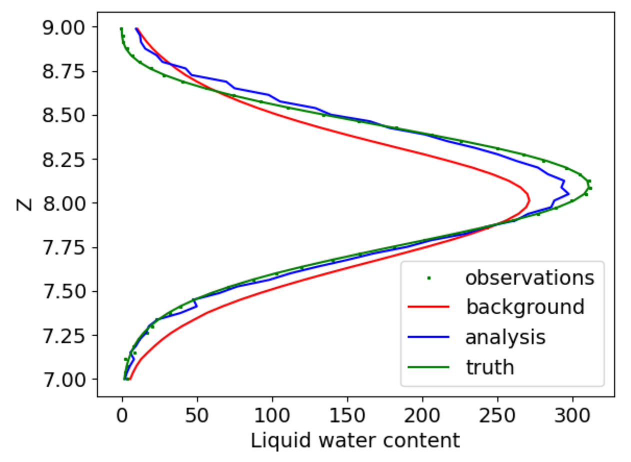
## DATA ASSIMILATION EXPERIMENTS

Data assimilation experiments are carried out, using

- **Truth run (toy model or reference model from [1] with Gaussian noise)**
- **Time evolution of model using two-moment-scheme from [1]**
- **DA employing EnKF or LETKF**




Truth generated by toy model. DA results with LETKF.




Truth generated by reference model of [1]. DA results with LETKF.

## Results

- Distinction between deterministic and stochastic quantities allows for calculation of expectation values and error covariance matrices
- In simplified setup: Large scale structure of covariances determined by gravitational sorting, small scale structure by hydrometeor geometry
- Provision of model with known error covariances and non-Gaussian errors for DA

## FUTURE RESEARCH

Topics of interest for future research include

- Improvement of DA experiments
- Generalization to more realistic dynamics (possibly stochastic); interactions between and creation and destruction of particles
- Inclusion of observation operators
- Modelling e.g., of rhiming by going from indicator functions to finitely supported density functions
- Searching for analytic solutions to obtained equations
- Link to (and perhaps application of) existing mathematical theory

**References:**

[1] Wacker, U. and A. Seifert (June 2001). "Evolution of rain water profiles resulting from pure sedimentation: Spectral vs. parameterized description". In: Atmospheric Research - ATMOS RES 58, pp. 19–39. DOI: 10.1016/S0169-8095(01)00081-3.

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