Formulation of Temporal Mean Flows with Averaging

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joint work with Beth Wingate and Terry Haut

Introduction

The Rotating Shallow Water Equations and the Boussinesq approximation are of particular interest for studying geophysical fluid dynamics. These problems admit the following general form

$$\frac{d\mathbf{u}}{dt} + \frac{1}{\varepsilon}\mathcal{L}\mathbf{u} = \mathcal{N}(\mathbf{u}). \quad (1)$$

Equation (1) is a partial differential equation and the vector $\mathbf{u}$ contains the variables of interest. The linear operator $\mathcal{L}$ has purely imaginary eigenvalues and makes the problems oscillatorily stiff. Additionally, the non-linearity $\mathcal{N}$ is bi-linear in fluid applications. The following transformation is central for our investigations

$$\mathbf{w}(t) = \exp\left(\frac{\mathcal{L}}{\varepsilon}t\right)\mathbf{u}(t). \quad (2)$$

One possibility to make use of the transformation is to eliminate the linear term in the time evolution problem (1) by applying the transformation (2). Applying averaging to further mitigate the oscillatory stiffness leads to the new problem

$$\frac{d\bar{\mathbf{w}}}{dt} = \frac{1}{\eta}\int_{-\eta/2}^{\eta/2} \rho\left(\frac{s}{\eta}\right) \exp\left(\frac{L}{\epsilon}(s+t)\right) \mathcal{N}\left(\exp\left(-\frac{L}{\epsilon}(s+t)\right)\bar{\mathbf{w}}(t)\right) ds. \quad (3)$$

Solving equation (3) can be very useful when parallel-in-time methods, like the Parareal method, shall be applied. From a fluid dynamics point of view, the averaging provides an interesting tool to formulate mean flows. Another possibility is to apply the transformation (2) and averaging to a given solution $\mathbf{u}(t)$ of problem (1) as a diagnostic tool and investigate if such formulations provide a suitable formulation for mean flows. In that case we evaluate

$$\bar{\mathbf{w}}(t) = \frac{1}{\eta}\int_{-\eta/2}^{\eta/2} \rho\left(\frac{s}{\eta}\right)\mathbf{w}(t+s)ds. \quad (4)$$

The 2D Boussinesq Approximation

$$u_t + uu_x + wu_z + \frac{\partial}{\partial x}\left(\Delta^{-1}\left(-\nabla\cdot(u\cdot\nabla u) - N\frac{\partial\bar{\rho}}{\partial z}\right)\right) = \nu\Delta u + F_u(x), \quad (5)$$

$$w_t + uw_x + ww_z + \frac{\partial}{\partial z}\left(\Delta^{-1}\left(-\nabla\cdot(u\cdot\nabla u) - N\frac{\partial\bar{\rho}}{\partial z}\right)\right) + N\bar{\rho}e_3 = \nu\Delta w + F_w(x), \quad (6)$$

$$\frac{\partial\bar{\rho}}{\partial t} + u\bar{\rho}_x + w\bar{\rho}_z - Nw = \kappa\Delta\bar{\rho}. \quad (7)$$

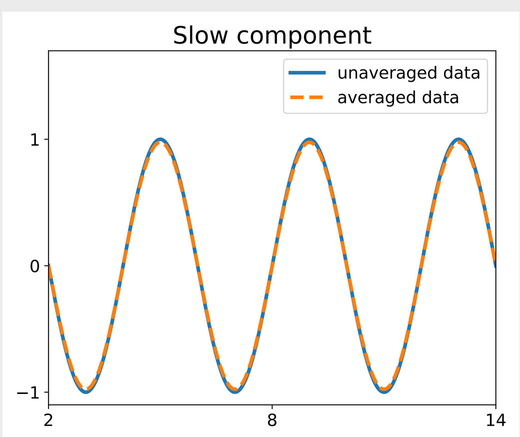
We assume periodic boundary conditions. The divergence constraint is given by $div\ \mathbf{v} = 0$ where $\mathbf{v} = (u, w)^T$. The pressure obeys a Poisson equation.

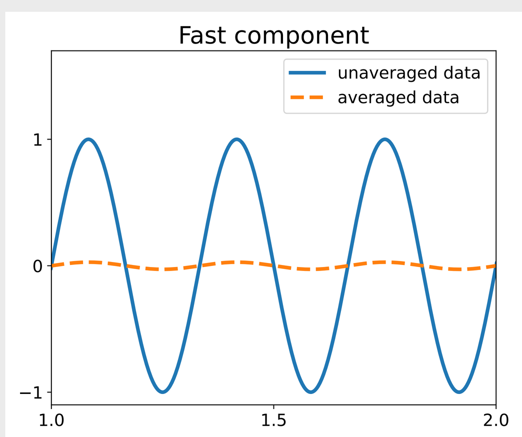
The Exponential and Averaging as a Diagnostic Tool

Motivation: In *Embid, Majda* (1998), the following identity was derived for a leading order solution of the rotating Boussinesq equations with strong stratification $u^0 = e^{-\tau\mathcal{L}}\bar{u}$, where $\bar{u}$ satisfies an averaged equation, which contains the non-linearity. Additionally, it is known from the aforementioned exposition that $\bar{u}$ can be decomposed into slow and fast components u^S and u^F. The slow component u^S evolves on a slow time scale, whereas the fast component u^F contains the fast oscillations. One possibility to define a mean flow u_{mean} is to set $u_{\text{mean}} = u^S$. With this definition we get rid of all fast oscillations. We will mitigate the constraint and allow some fast oscillations. The averaging procedure will be of central significance to decide which oscillations shall remain in the solution.

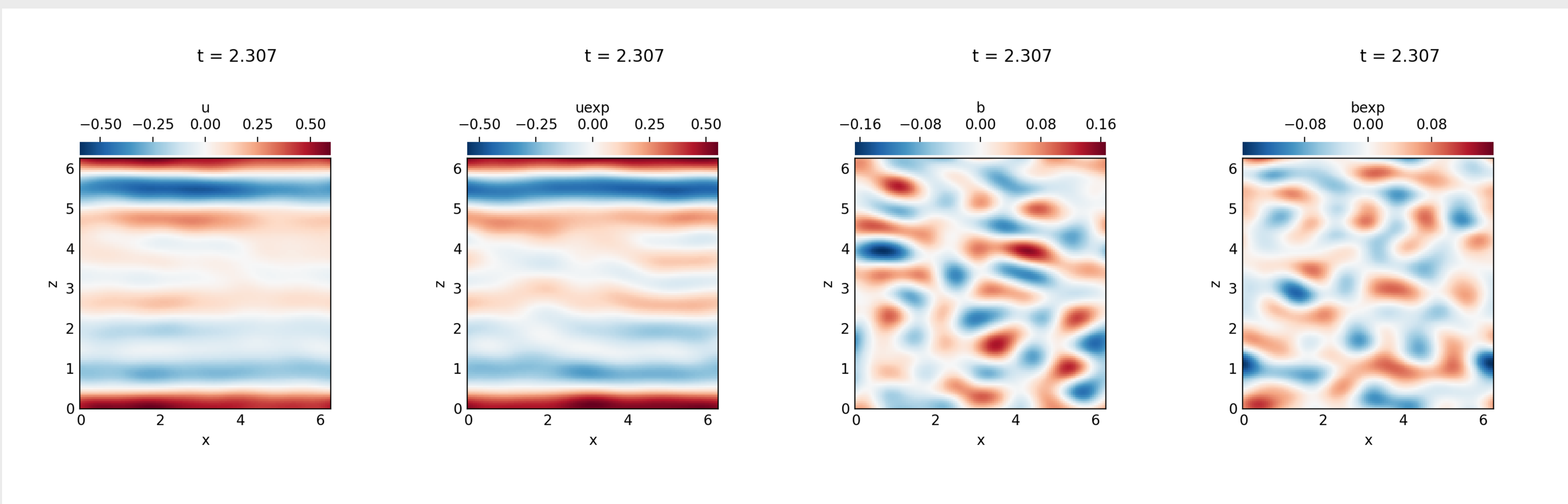
The Exponential of the Linear Operator: Implementing the exponential of a linear operator can be a challenge. We apply a Fourier decomposition in space. Then, the matrix exponentials for the different wave numbers are computed.

Averaging of Fast Modes: In the left figure, the averaging window η is smaller than the period of the oscillations. The data is barely altered. In the right figure, the averaging window η is larger than the period of the oscillations. Therefore, the oscillations are damped.





Application of the Exponential to Simulation Data



The 1D Rotating Shallow Water Equations

$$\frac{du}{dt} + \frac{1}{\epsilon}\left(-v + F^{-1/2}\frac{dh}{dx}\right) + u\frac{dv_1}{dx} = \mu\partial_x^4 u \quad (8)$$

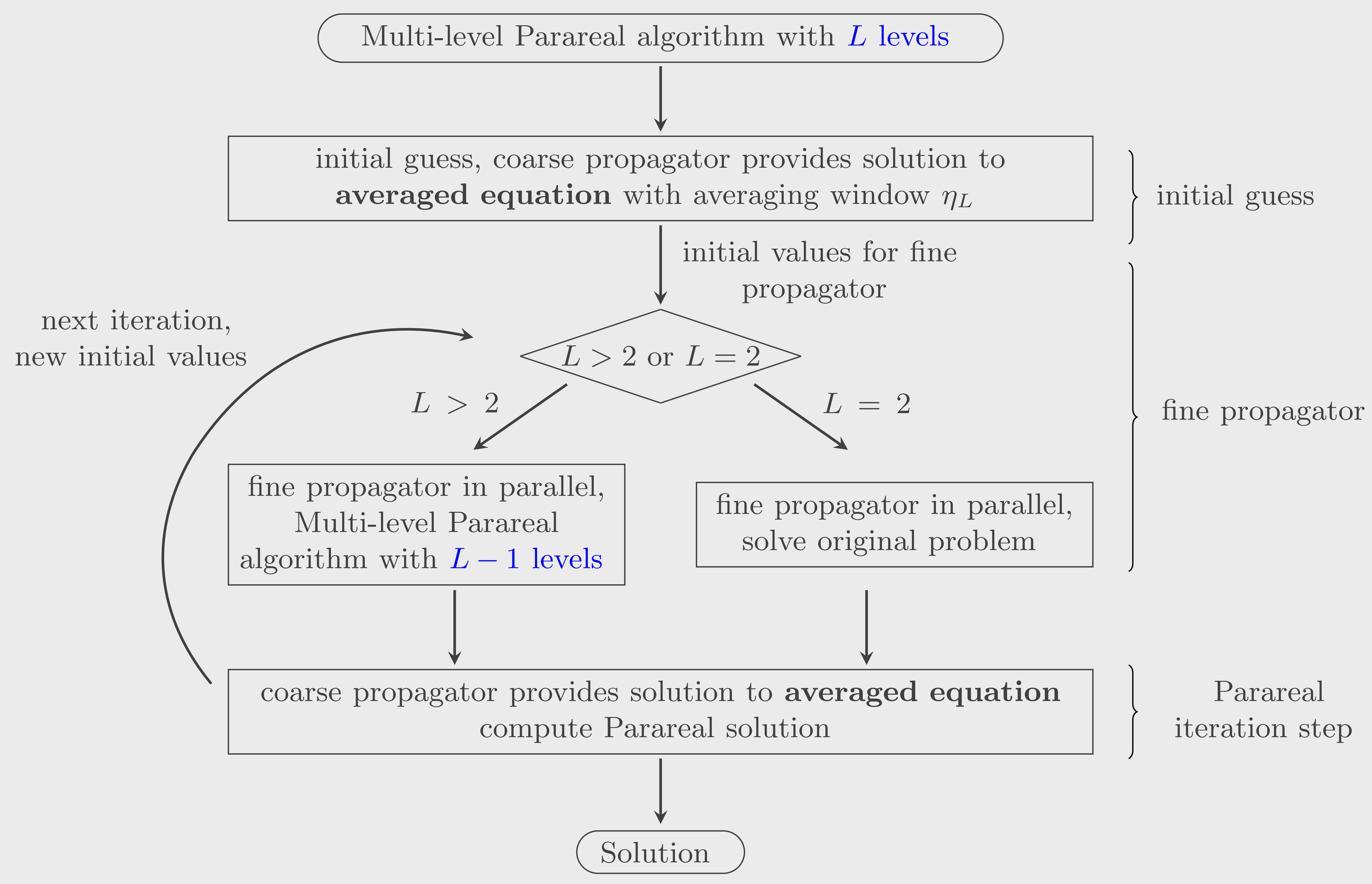
$$\frac{dv}{dt} + \frac{1}{\epsilon}u + u\frac{dv}{dx} = \mu\partial_x^4 v \quad (9)$$

$$\frac{dh}{dt} + \frac{F^{-1/2}}{\epsilon}\frac{du}{dx} + \frac{\partial}{\partial x}(hu) = \mu\partial_x^4 h, \quad (10)$$

where $(u, v)^T$ is the velocity and h denotes the height. Additionally, ϵ denotes the Rossby number and F is the Burger number. A hyperviscosity term is added to the right-hand side. The equations were solved efficiently with the Multi-level Parareal method with Averaging.

The Multi-level Parareal Method with Averaging

- Parallel-in-time method based on the Parareal method, has a coarse and a fine propagator
- Fine propagator can be applied in parallel
- Fine propagator of Multi-level Parareal Method with L levels is Multi-level Parareal Method with $L-1$ levels
- Uses formulation (3), but computes solution to the unaveraged problem
- Goal of parallel-in-time methods: faster time-to-solution compared to serial time-steppers, goal when using multiple levels: increasing the efficiency compared to the two-level case



source: J. Rosemeier, T. Haut, B. Wingate, Multi-level Parareal Algorithm with Averaging for Oscillatory Problems

References

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