Math 3A, Section B, Spring 1997

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Final Exam

1. Solve the system of linear equations

$$\begin{pmatrix} -1 & -4 & -4 & -9 \\ -2 & -8 & 8 & 30 \\ -3 & -12 & -1 & 6 \\ -4 & -16 & -2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ -14 \\ 5 \\ 8 \end{pmatrix}.$$

   Use the method taught in class. Check your answer (required!). (15)

2. Let $V = P_3$, the vector space of polynomials of degree less or equal to 3. Find a basis for V containing the vectors

$$\begin{aligned} v_1 &= 1 - 2x, \\ v_2 &= 1 + x + x^2 + x^3, \\ v_3 &= 1 - x^2 - x^3. \end{aligned}$$

   (10)

3. (a) Diagonalize the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{pmatrix}.$$

   I.e., find the eigenvalues λ_i of A, an invertible matrix C, and a diagonal matrix D such that $D = C^{-1}AC$.

   (b) Find A^{10}.

   (10+5)

4. (a) Find an orthonormal basis for

$$W = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} \right\}.$$

(b) Find the projection of $v = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ onto W.

(5+5)

5. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^4$ be defined by

$$T(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{pmatrix}.$$

Find the matrix $T_{B,E}$ which represents T when $\mathbb{R}^2$ is endowed with the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

and $\mathbb{R}^4$ is endowed with the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. (10)

6. (a) Is the set of all matrices for which the sum of the diagonal elements is 0, endowed with the usual addition and scalar multiplication, a vector space? Explain.

   (b) Is the set of all matrices for which the sum of the diagonal elements is 1, endowed with the usual addition and scalar multiplication, a vector space? Explain why this is different from the previous case.

(5+5)

7. A matrix $A \in M(n \times n)$ is called *nilpotent* if there exists a positive integer k such that $A^k = 0$. Prove that a nilpotent matrix is not invertible. (5)

8. Let V be the vector space of continuous functions on the interval $[0, 2\pi]$. Recall from Calculus that the mean (or average) of a

function is defined

$$M(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

for every $f \in V$.

(a) Prove that M is a linear transformation on V.

(b) Let W be a subspace of V with basis

$$B = \{1, \sin x, \cos x, \sin 2x, \cos 2x\}$$

and consider M restricted to W. Then

$$\dim W = \boxed{},$$
$$\dim \operatorname{range} M = \boxed{},$$
$$\dim \ker M = \boxed{}.$$

Fill in the blanks and explain *briefly* how you got your answer.

(c) Describe $\ker M$ as concretely as you can.

(5+5+5)

9. Let $A \in M(n \times n)$ satisfy the equation

$$AA^T = I.$$

(Such a matrix is called *orthogonal*.)

(a) Is A invertible? Explain.

(b) Prove that if λ is an eigenvalue of A, then either $\lambda = 1$ or $\lambda = -1$.
Hint: Compare the eigenvalues of A with those of A^{-1}.

(5+5)