Übung 6 – Lösungen

1. Wir müssen unabhängige Variationen in q und p nehmen. Also:

$$0 = \delta \mathcal{W}_H = \int_{t_1}^{t_2} \left[\langle \delta p, \dot q \rangle + \langle p, \delta \dot q \rangle - H_q \delta q - H_p \delta p \right] dt$$

$$\overset{\text{P.I.}}{=} \int_{t_1}^{t_2} \langle \dot q - H_p, \delta p \rangle \, dt + \int_{t_1}^{t_2} \langle -\dot p - H_q, \delta q \rangle \, dt$$

Nach dem Lemma der Variationsrechnung:

$$\left. \begin{aligned} \dot q - H_p &= 0 \\ \dot p + H_q &= 0 \end{aligned} \right\} (*)$$

3. Schreibe $(*)$ als

$$\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = J \nabla H \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

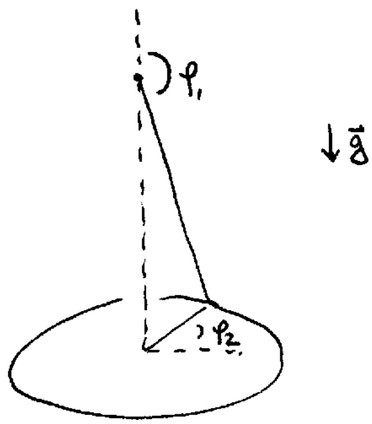
$$\nabla = \begin{pmatrix} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{pmatrix}$$

$$\Rightarrow \frac{d}{dt} H(q,p) \overset{\text{Kettenregel}}{=} \left\langle \nabla H(q,p), \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} \right\rangle$$

$$= \langle \nabla H(q,p), J \nabla H(q,p) \rangle$$

$$= 0, \text{ da } J \text{ antisymmetrisch.}$$

2.



$$E_{pot} = + mg\ell \cos \varphi_1$$

$$E_{kin} = \tfrac{1}{2} m \left(\ell \dot{\varphi}_1\right)^2 + \tfrac{1}{2} m \left(\ell (\sin \varphi_1)\, \dot{\varphi}_2\right)^2$$

$$= \tfrac{1}{2} m \ell^2 \left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 \sin^2 \varphi_1 \right)$$

$$L = E_{kin} - E_{pot}$$

$$\frac{\partial L}{\partial \dot{\varphi}_1} = m \ell^2 \dot{\varphi}_1 \qquad \frac{\partial L}{\partial \dot{\varphi}_2} = m \ell^2 \dot{\varphi}_2 \sin^2 \varphi_1$$

$$\frac{\partial^2 L}{\partial \dot{\varphi}_1^2} = m \ell^2 > 0 \qquad \frac{\partial^2 L}{\partial \dot{\varphi}_1 \partial \dot{\varphi}_2} = 0 \qquad \frac{\partial^2 L}{\partial \dot{\varphi}_2^2} = m \ell^2 \sin^2 \varphi_1$$

$\Rightarrow$ $\mathcal{H}ess_{\dot{\varphi}} L$ ist pos. def. falls $\varphi_1 \in \,]0, \pi[$.

Die Legendre-Transformation bildet ab

$$(\varphi, \dot{\varphi}) \mapsto (\varphi, \nabla_{\dot{\varphi}} L) = \left(\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} m \ell^2 \dot{\varphi}_1 \\ m \ell^2 \dot{\varphi}_2 \sin^2 \varphi_1 \end{pmatrix} \right)$$

$$\Rightarrow \Omega_H = \,]0, \pi[\times [0, 2\pi[\times \mathbb{R}^2 = \Omega_L$$

$$p_1 = \frac{\partial L}{\partial \dot\varphi_1} = m\ell^2 \dot\varphi_1 \Rightarrow \dot\varphi_1 = \frac{p_1}{m\ell^2}$$

$$p_2 = \frac{\partial L}{\partial \dot\varphi_2} = m\ell^2 \dot\varphi_2 \sin^2\varphi_1 \Rightarrow \dot\varphi_2 = \frac{p_2}{m\ell^2 \sin^2\varphi_1}$$

$$H = \langle \dot\varphi, p \rangle - L(\varphi, \dot\varphi)$$

$$= \left(\frac{p_1^2}{m\ell^2} + \frac{p_2^2}{m\ell^2 \sin^2\varphi_1} \right) - \frac{1}{2} m\ell^2 \left(\left(\frac{p_1}{m\ell^2}\right)^2 + \left(\frac{p_2}{m\ell^2 \sin^2\varphi_1}\right)^2 \sin^2\varphi_1 \right)$$

$$+ mg\ell \cos\varphi_1$$

$$= \frac{1}{2} \left(\frac{p_1^2}{m\ell^2} + \frac{p_2^2}{m\ell^2 \sin^2\varphi_1} \right) + mg\ell \cos\varphi_1$$

Hamilton'sche Bewegungsgleichungen:

$$\dot\varphi_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m\ell^2}$$

$$\dot\varphi_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m\ell^2 \sin^2\varphi_1}$$

$$\dot p_1 = -\frac{\partial H}{\partial \varphi_1} = \frac{-p_2^2}{2m\ell^2} \cos\varphi_1 \left(-2 \sin^{-3}\varphi_1\right) + mg\ell \sin\varphi_1$$

$$= \frac{p_2^2}{m\ell^2} \frac{1}{\sin^2\varphi_1 \tan\varphi_1} + mg\ell \sin\varphi_1$$

$$\dot p_2 = -\frac{\partial H}{\partial p_2} = 0$$