Solutions Week 3 - For discussion Oct. 30, 2025

2.13 The rank-nullity theorem for a linear map $K: W \to X$ says:

$$\dim W = \dim R(K) + \dim N(K) \qquad (***)$$

Now restrict the map K to the subspace $R(L) \subset W$, where L is a linear map $L: V \to W$.

By construction, $R(KL) = R(K|_{R(L)})$. But for $K|_{R(L)}$, the rank-nullity theorem says

$$\dim R(L) = \dim R(K|_{R(L)}) + \dim (N(K) \cap R(L))$$

$$\Rightarrow \dim R(KL) = \dim R(L) - \dim (N(K) \cap R(L)) \qquad (*)$$

2.14 (i) From (*): $\dim R(KL) \leq \dim R(L)$

Moreover, $R(KL) \subset R(K)$ because if $x \in R(KL)$,

$\exists\, v \in V$ s.t. $x = KLv \Rightarrow x = Kw$ with $w = Lv \Rightarrow x \in R(K)$

$\Rightarrow \dim R(KL) \leq \dim R(K)$

So altogether: $\dim R(KL) \leq \min\{\dim R(K), \dim R(L)\}$

(ii) Clearly, $\dim N(K) \geq \dim (N(K) \cap R(L))$

From (***): $\dim R(KL) \geq \dim R(L) - \dim N(K)$

$\overset{(***)}{=} \dim R(L) + \dim R(K) - \dim W$

221 $S = [e^{i\theta}, e^{-i\theta}]$, $T = [\cos\theta, \sin\theta]$

(i) $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$ (*)

$$\Rightarrow M_{TS}^{Id} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

(ii) First option: $M_{ST}^{Id} = \left(M_{TS}^{Id}\right)^{-1}$, so need to compute inv.:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ i & -i & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2i & -i & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2i} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2i} \end{array}\right)$$

$$\Rightarrow M_{ST}^{Id} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Second option: From (*), we easily get

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow M_{ST}^{Id} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$ as before

(iii) By direct computation...

2.23 $L(x) = Ax = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_k \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$

$= x_1 a_1 + \cdots + x_k a_k$ by rules of matrix multiplication.

$\Rightarrow L(x) \in \operatorname{span}\{a_1, \ldots, a_k\}$

vice versa, if $y \in \operatorname{span}\{a_1, \ldots, a_k\}$, the computation above shows that $y = L(x)$ where the elements of x are the coefficients of the l.c.

2.30 Suppose $\mathbb{F}^m \cong \mathbb{F}^n$. Then there is a bijective map $L: \mathbb{F}^m \to \mathbb{F}^n$. Since L injective, $N(L) = \{0\}$.
Since L surjective, $R(L) = \mathbb{F}^n$.

Then the rank-nullity theorem reads:

$$\underbrace{\dim \mathbb{F}^m}_{= m} = \underbrace{\dim R(L)}_{= n} + \underbrace{\dim N(L)}_{= 0}$$

$\Rightarrow m = n$.

Solutions - Week 3, Homework

2.19 Let's first use the basis $T = [1, x, x^2]$.

$$L[1] = 1 + 4 \cdot 1' = 1$$

$$L[x] = x + 4x' = 4 + x$$

$$L[x^2] = x^2 + 4(x^2)' = 8x + x^2$$

$$\Rightarrow M_{TT}^{L} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}$$

By inspection, we see that, with $S = [x^2+1, x-1, 1]$

$$M_{TS}^{Id} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Need to compute the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$= M_{ST}^{Id}$$

$$\Rightarrow M_{SS}^{L} = M_{ST}^{Id} \, M_{TT}^{L} \, M_{TS}^{Id} = M_{ST}^{Id} \begin{pmatrix} 1 & 3 & 1 \\ 8 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 4 & 1 \end{pmatrix}$$

Alternative solution:

$$L[x^2+1] = x^2+1 + 4\cdot 2x$$

$$= 8(x-1) + 8 + (x^2+1) - 1 + 1$$

$$= 8s_3 + 8s_2 + s_1$$

$$L[x-1] = x-1 + 4\cdot 1 = (x-1) + 4 = s_2 + 4s_3$$

$$L[1] = 1 + 4\cdot 0 = s_3$$

$$\Rightarrow M_{SS}^{L} = \begin{pmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 4 & 1 \end{pmatrix}$$

(The alternative solution looks easier, but this is only due to the fact that the expansion of $L[s_i]$ in terms of s_1, s_2, s_3 is particularly easy to spot without much computation.)

2.20 $S = [e^{\alpha x}, xe^{\alpha x}, x^2e^{\alpha x}]$

$$D[s_1] = (e^{\alpha x})' = \alpha e^{\alpha x} = \alpha s_1$$

$$D[s_2] = (xe^{\alpha x})' = e^{\alpha x} + \alpha x e^{\alpha x} = s_1 + \alpha s_2$$

$$D[s_3] = (x^2e^{\alpha x})' = 2xe^{\alpha x} + \alpha x^2 e^{\alpha x} = 2s_2 + \alpha s_3$$

$$\Rightarrow M_{SS}^{D} = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 2 \\ 0 & 0 & \alpha \end{pmatrix}$$

2.25 $A = (a_{ij})$ $\quad B = (b_{ij})$

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} \qquad (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

(i) $$\operatorname{tr} AB = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} = \operatorname{tr} BA$$

(ii) A, B similar $\Rightarrow \exists S$ invertible s.t. $A = SBS^{-1}$

$$\Rightarrow \operatorname{tr} A = \operatorname{tr}(SBS^{-1}) \overset{(i)}{=} \operatorname{tr}(S^{-1}SB) = \operatorname{tr}(IB) = \operatorname{tr} B$$

(iii) $$\operatorname{tr} AB^T = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ik} = \operatorname{tr} B^T A$$

by direct application of the definition of trace, matrix product, and transpose.

Notes:

- $\langle A, B \rangle = \operatorname{tr}(A^H B)$ defines an inner product ("Frobenius inner product") that is often used in matrix optimization problems.
- (i) implies that $\operatorname{tr}(A_1 \cdots A_k) = \operatorname{tr}(A_2 \cdots A_k A_1)$
  $= \operatorname{tr}(A_3 \cdots A_k A_1 A_2)$
  $= \cdots$

  I.e., the trace is invariant under cyclic permutations of matrix products.
- (i) remains true if $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times m}(\mathbb{F})$

  (iii) remains true if $A, B \in M_{m\times n}(\mathbb{F})$

2.38 (i) $(e_k)_i = \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}$ (Standard unit vector)

$$\Rightarrow \left(e_k e_\ell^T\right)_{ij} = \begin{cases} 1 & \text{if } i=k,\ j=\ell \\ 0 & \text{otherwise} \end{cases}$$

(ii) Direct proof – check the property of the inverse:

(Note that, WLOG, we can take $a=1$ if we do not require that u, v are unit vectors.)

$$\left(I - uv^T\right)\left(I - \frac{uv^T}{v^Tu - 1}\right) = I - uv^T - \frac{uv^T}{v^Tu-1} + \frac{uv^Tuv^T}{v^Tu-1}$$

$$= I - uv^T\left[1 + \frac{1}{v^Tu-1} - \frac{v^Tu}{v^Tu-1}\right]$$

$$= \frac{v^Tu - 1 + 1 - v^Tu}{v^Tu-1} = 0$$

$$= I$$

Remark: This is a special case of the Sherman-Morrison formula

$$\left(A + uv^T\right)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$$

For a derivation, see Advanced Programming class notes from Winter 2024/25, Appendix A4.

2.40 B is row equivalent to A if $B = E_1 \cdots E_k A$ where $E_1, \ldots, E_k$ are elementary matrices.

(i) $A = \underset{\uparrow}{I} A$, so A is row-equivalent to itself.
elementary matrix

(ii) If B is row equivalent to A, then

$$A = E_k^{-1} \cdots E_1^{-1} B$$

where E_i^{-1} is also an elementary matrix, so A is row equivalent to B

(iii) If A is row-equivalent to B and B is row equivalent to C, then

$$\left.\begin{aligned} A &= E_1 \cdots E_k B \\ B &= E_{k+1} \cdots E_p C \end{aligned}\right\} \quad A = E_1 \cdots E_p C$$

$\Rightarrow$ A is row-equivalent to C.