Week 2 - Solution to exercises

2.1 (i) $L(x,y) = (y,x)$ is linear, because:

$$L(ax_1 + bx_2,\ ay_1 + by_2) = (ay_1 + by_2,\ ax_1 + bx_2)$$
$$= a(y_1, x_1) + b(y_2, x_2)$$
$$= a\,L(x_1, y_1) + b\,L(x_2, y_2)$$

alternatively $L(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

and we know that matrix multiplication defines a linear transformation in this way.

$N(L) = \{0\}$ (because $(y,x)=0$ implies $x=0$ and $y=0$)
$R(L) = \mathbb{R}^2$ (by inspection)

(ii) $L(x,y) = (x,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is linear, cf. (i)

$N(L) = \text{span}\{e_2\}$
$R(L) = \text{span}\{e_1\}$

(iii) Not linear, as $L(0,0) = (0+1, 0+1) = (1,1) \neq 0$

(iv) Not linear, as $L(ax, ay) = ((ax)^2, (ay)^2) = a^2(x^2, y^2)$
$\neq a\,L(x,y) = a(x^2, y^2)$

unless $a = \pm 1$.

2.2 $V = (0,\infty)$ with $x \oplus y = xy$

$a \odot x = x^a$

$T(x) = \log x$ maps $(0,\infty)$ into $\mathbb{R}$.

$$T(a\odot x \oplus b \odot y) = \log(x^a y^b)$$
$$= a\log x + b \log y$$
$$= aT(x) + bT(y)$$

So T is linear in the sense specified.

2.3 (i) $L[p](x) = x^2$ is not linear, because $L[0](x) = x^2 \neq 0$

(ii) $L[p](x) = x\,p(x)$ is linear, as

$$L[ap+bq](x) = x(ap(x) + bq(x))$$
$$= a\,x\,p(x) + b\,x\,q(x)$$
$$= a\,L[p](x) + b\,L[q](x)$$

Note: Setting $S = [1, x, x^2]$ and $T = [1, x, x^2, x^3, x^4]$,

$$M_{TS}^{L} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$N(L) = \{0\}$ $\quad R(L) = \text{span}\{x, x^2, x^3\}$

(iii) $L[p](x) = x^4 + p(x)$

is not linear, because $L[0](x) = x^4 + 0 \neq 0$

(iv) $L[p](x) = (4x^2 - 3x)\,p'(x)$ is linear, as

$$L[ap+bq](x) = (4x^2-3x)\,(ap+bq)'(x)$$
$$= (4x^2-3x)\,(a\,p'(x) + b\,q'(x))$$
$$= a\,(4x^2-3x)\,p'(x) + b\,(4x^2-3x)\,q'(x)$$
$$= a\,L[p](x) + b\,L[q](x)$$

Note: $L[1](x) = 0$

$L[x](x) = (4x^2-3x)\cdot 1 = 4x^2 - 3x$

$L[x^2](x) = (4x^2-3x)(2x) = 8x^3 - 6x^2$

$$\Rightarrow \quad M_{TS}^{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 4 & -6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$N(L) = \operatorname{span}\{1\}$

$R(L) = \operatorname{span}\{4x^2-3x,\ 8x^3-6x^2\}$

Solutions - Week 2 - written homework

1.9 Given: $v \in \text{span}\{v_1, \ldots, v_n\} = V$

$\Rightarrow \exists\ \alpha_1, \ldots, \alpha_n \in \mathbb{F}$ s.t. $\quad v = \alpha_1 v_1 + \ldots + \alpha_n v_n$

$\Rightarrow 0 = \alpha_0 v + \alpha_1 v_1 + \ldots + \alpha_n v_n \quad$ with $\alpha_0 = -1$

$\Rightarrow$ 0 is represented by a nontrivial l.c. of vectors $\{v, v_1, \ldots v_n\}$

$\Rightarrow \{v, v_1, \ldots, v_n\}$ is l.d.

1.11 Since $\mathbb{F}[x]$ is defined as the set of all linear combinations of elements from $\{1, x, x^2, \ldots\}$, we only need to check that they are l.i.. Suppose that

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k = 0 \qquad (*)$$

(i) Set $x = 0 \Rightarrow a_0 = 0$

(ii) Differentiate (*):

$$a_1 + 2a_2 x + \ldots + k a_k x^{k-1} = 0$$

(iii) Set $x = 0 \Rightarrow a_1 = 0$

Now repeat argument from (ii). After k steps, we see that

$$a_0 = a_1 = \ldots = a_k = 0$$

$\Rightarrow \{1, x, \ldots, x^k\}$ is l.i.

Since this is true for any k, we conclude that $\{1, x, x^2, \ldots\}$ is l.i.

1.12 $\{v_1, \ldots, v_n\}$ l.i. $\Rightarrow$ $n \leq \dim V$ (Theorem from class)

$\Rightarrow \{v_2, \ldots, v_n\}$ has strictly less than $\dim V$ elements

$\Rightarrow$ it cannot span V (same Theorem from class, cf. Cor. 1.4.4)

1.14 Given: any vector in V can be uniquely expressed as a lc. of elements from S,

$\Rightarrow$ span $S = V$

and S is l.i. (since also the representation of 0 is unique)

$\Rightarrow$ S is basis.

1.23 Given: S is l.i., $|S| = \dim V$

Suppose $\exists\ v \notin \text{span}\, S$. Then $\{v\} \cup S$ is l.i. and has $\dim V + 1$ elements. So it cannot be l.i. $\Rightarrow$ contradiction

$\Rightarrow$ S is basis.