

Exercise 3 - Solutions

$$\begin{array}{r} 1. \quad 10001011 \\ \quad 11111000 \\ \quad 00000000 \\ \text{XOR } 01011010 \\ \hline \quad 00101001 \end{array} \quad \leftarrow \text{Reconstructed bit sequence on failed disk}$$

2. First, we prove the hint.

$$\text{Given: } a \equiv a' \pmod{97}$$

This means that $a = a' + n \cdot 97$ for some $n \in \mathbb{Z}$

$$\begin{aligned} \Rightarrow 10^k a + b &= 10^k a' + 10^k \cdot n \cdot 97 + b \\ &= 10^k a' + b + m \cdot 97 \quad \text{for some } m \in \mathbb{Z} \end{aligned}$$

$$\Rightarrow 10^k a + b \equiv 10^k a' + b \pmod{97}$$

Now we want to compute $i \pmod{97}$ for some big integer i .

$$\text{Fix } k \text{ and write } i = 10^k a + b.$$

$\uparrow \quad \quad \uparrow$
all digits up to decimal digit $k+1$ last k digits of i

$$(\text{E.g. } k=3, i=286793 \text{ can be written } i = 10^3 \cdot 286 + 793.)$$

Then compute $a' := a \pmod{97}$, i.e. a' has at most two decimal digits.

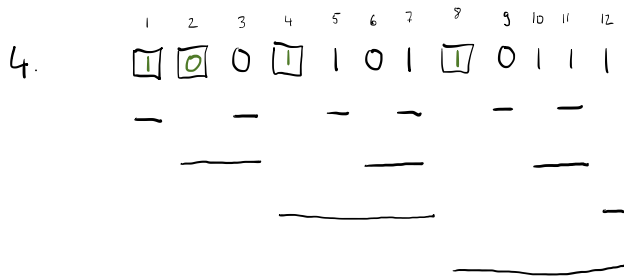
$$\begin{aligned} \text{Then, by the hint, } i &\equiv \underbrace{10^k a'}_=: i' + b \pmod{97} \\ &=: i' \end{aligned}$$

So we can continue to compute $i' \pmod{97}$, where i' is potentially shorter than i as it has at most $k+2$ decimal digits.

$$3(a). \quad \text{DE5175...} \rightarrow 7503030000763300 \underbrace{131451}_{\substack{\text{DE} \\ \text{D E}}} =: i \quad \text{Now } i \equiv 1 \pmod{97}, \text{ so code is } \underline{\text{valid}}$$

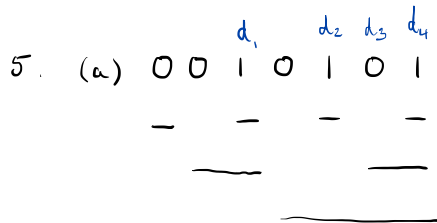
(In Python, compute $i \% 97$)

$$(b) \quad \text{Similarly, } 750303000007363300131451 = 36 \pmod{97}, \text{ so code is } \underline{\text{invalid}}.$$



parity bits green

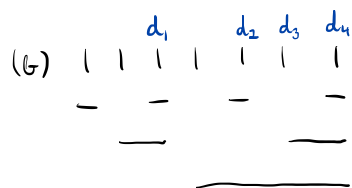
data bits black



$$\begin{cases} p_1 = 1 \\ p_2 = 0 \\ p_4 = 0 \end{cases}$$

There is an error at position $(001)_2 = 1$

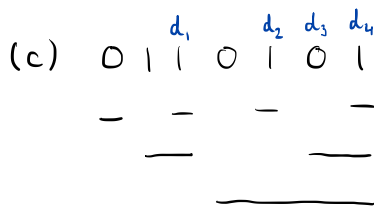
The correct message is 1101



$$\begin{cases} p_1 = 0 \\ p_2 = 0 \\ p_4 = 0 \end{cases}$$

The code is correct.

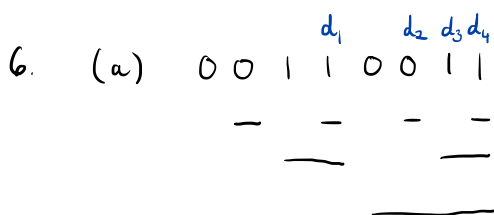
The correct message is 1111



$$\begin{cases} p_1 = 1 \\ p_2 = 1 \\ p_4 = 0 \end{cases}$$

The code has an error at $(011)_2 = 3$

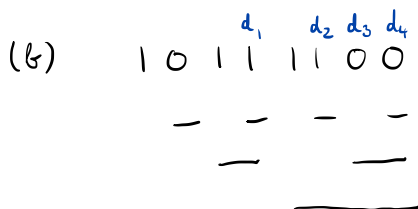
The correct message is 0101



$p_0 = 0$, so overall parity is O.K.

$$\begin{cases} p_1 = 0 \\ p_2 = 0 \\ p_4 = 0 \end{cases}$$

Code correct, message is 1011

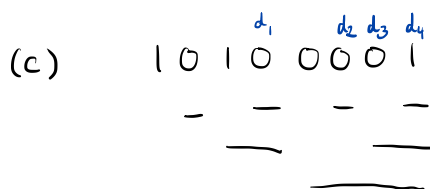


$p_0 = 1$, assume single-bit error

$$\begin{cases} p_1 = 0 \\ p_2 = 0 \\ p_4 = 0 \end{cases}$$

all O.K., so single-bit error must be in p_0

Message is 1100



$p_0 = 1$, assume single-bit error

$$\begin{cases} p_1 = 1 \\ p_2 = 0 \\ p_4 = 1 \end{cases}$$

Error at $(101)_2 = 5$ (at data bit d_2)

Message is 0101

(d) 0 0 1 1 0 1 0 1 $p_0 = 0$, so overall parity O.K.

—	—	—	—	$\left. \begin{array}{l} p_1 = 1 \\ p_2 = 1 \\ p_4 = 0 \end{array} \right\}$	indicates error, which must be 2-bit Discard message.
—	—	—	—		
—	—	—	—		

7. The errors follow a binomial distribution, with $p = \frac{1}{100}$, $q = 1-p = \frac{99}{100}$

(a) $P(0 \text{ errors in 4 bits}) = \binom{4}{0} p^0 q^{4-0} = \left(\frac{99}{100}\right)^4 \approx 0.961$

(b) $P(0 \text{ or } 1 \text{ error in 7 bits}) = \binom{7}{0} p^0 q^{7-0} + \binom{7}{1} p^1 q^{7-1} \approx 0.9980$

(c) Assume that re-transmission is 100% successful, i.e. a detected two-bit error will result in a correct message. Since the true value is ~~very~~ close to 100%, the error in this approximation is practically negligible. So we need to compute:

$P(0, 1 \text{ or } 2 \text{ errors in 8 bits}) = \binom{8}{0} p^0 q^{8-0} + \binom{8}{1} p^1 q^{7-1} + \binom{8}{2} p^2 q^{7-2} \approx 0.99997$