Exercise 9 - Solutions

1.

$$
\begin{array}{r l}
 & 1000\,1011 \\
 & 1111\,1000 \\
 & 0000\,0000 \\
\text{XOR} & 0101\,1010 \\
\hline
 & 0010\,1001
\end{array}
$$

$\leftarrow$ Reconstructed bit sequence on failed disk

2. First, we prove the hint.

Given: $a \equiv a' \mod 97$

This means that $a = a' + n \cdot 97$ for some $n \in \mathbb{Z}$

$$\Rightarrow \quad 10^k a + b = 10^k a' + 10^k \cdot n \cdot 97 + b$$

$$= 10^k a' + b + m \cdot 97 \qquad \text{for some } m \in \mathbb{Z}$$

$$\Rightarrow \quad 10^k a + b \equiv 10^k a' + b \mod 97$$

Now we want to compute $i \bmod 97$ for some big integer i.

Fix k and write $\quad i = 10^k a + b$.

b: last k digits of i

a: all digits up to decimal digit $k+1$

(E.g. $k=3$, $i = 286793$ can be written $i = 10^3 \cdot 286 + 793$.)

Then compute $a' := a \bmod 97$, i.e. a' has at most two decimal digits.

Then, by the hint, $\quad i \equiv \underbrace{10^k a' + b}_{=:\, i'} \mod 97$

So we can continue to compute $i' \bmod 97$, where i' is potentially shorter than i as it has at most $k+2$ decimal digits.

3(a). DE51 75... $\rightarrow$ $75090300000763330 0\underbrace{13}_{D}\underbrace{14}_{E}51 =: i$ Now $i \equiv 1 \mod 97$, so code is <u>valid</u>

(In Python, compute `i % 97`)

(b) Similarly, $750903000007363300131451 \equiv 36 \mod 97$, so code is <u>invalid</u>.

4.

1	2	3	4	5	6	7	8	9	10	11	12
$\boxed{1}$	$\boxed{0}$	0	$\boxed{1}$	1	0	1	$\boxed{1}$	0	1	1	1

parity bits green

data bits black

5. (a)

$$\begin{array}{ccccccc} & & d_1 & & d_2 & d_3 & d_4 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

$p_1 = 1$
$p_2 = 0$
$p_4 = 0$

There is an error at position $(001)_2 = 1$

The correct message is 1101

(b)

$$\begin{array}{ccccccc} & & d_1 & & d_2 & d_3 & d_4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

$p_1 = 0$
$p_2 = 0$
$p_4 = 0$

The code is correct.

The correct message is 1111

(c)

$$\begin{array}{ccccccc} & & d_1 & & d_2 & d_3 & d_4 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}$$

$p_1 = 1$
$p_2 = 1$
$p_4 = 0$

The code has an error at $(011)_2 = 3$

The correct message is 0101

6. (a)

$$\begin{array}{cccccccc} & & & d_1 & & d_2 & d_3 & d_4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}$$

$p_0 = 0$, so overall parity is O.K.

$p_1 = 0$
$p_2 = 0$
$p_4 = 0$

Code correct, message is 1011

(b)

$$\begin{array}{cccccccc} & & & d_1 & & d_2 & d_3 & d_4 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}$$

$p_0 = 1$, assume single-bit error

$p_1 = 0$
$p_2 = 0$
$p_4 = 0$

all O.K., so single-bit error must be in p_0

Message is 1100

(c)

$$\begin{array}{cccccccc} & & & d_1 & & d_2 & d_3 & d_4 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$p_0 = 1$, assume single-bit error

$p_1 = 1$
$p_2 = 0$
$p_4 = 1$

Error at $(101)_2 = 5$ (at data bit d_2)

Message is 0101

(d) 0 0 1 1 0 1 0 1 $\quad p_0 = 0$, so overall parity O.K.

$$\left.\begin{array}{l} p_1 = 1 \\ p_2 = 1 \\ p_4 = 0 \end{array}\right\}$$ indicates error, which must be 2-bit

Discard message.

7. The errors follow a binomial distribution, with $p = \frac{1}{100}$, $q = 1-p = \frac{99}{100}$

(a) $P(0 \text{ errors in } 4 \text{ bits}) = \binom{4}{0} p^0 \cdot q^{4-0} = \left(\frac{99}{100}\right)^4 \approx 0.961$

(b) $P(0 \text{ or } 1 \text{ errors in } 7 \text{ bits}) = \binom{7}{0} p^0 q^{7-0} + \binom{7}{1} p^1 q^{7-1} \approx 0.9980$

(c) Assume that re-transmission is 100% successful, i.e. a detected two-bit error will result in a correct message. Since the true value is very close to 100%, the error in this approximation is practically negligible. So we need to compute:

$P(0,1 \text{ or } 2 \text{ errors in } 8 \text{ bits}) = \binom{8}{0} p^0 q^{8-0} + \binom{8}{1} p^1 q^{7-1} + \binom{8}{2} p^2 q^{7-2} \approx 0.99997$