Solutions – Discussion Week 9

5.36 $\quad f_n(x) = \sin^n(x)$ on the interval $[0,\pi]$

(i) If $x = \frac{\pi}{2}$, $\sin(x) = 1 \Rightarrow f_n(x) = 1 \xrightarrow[n\to\infty]{} 1$

If $x \neq \frac{\pi}{2}$, $\sin(x) \in [0,1) \Rightarrow f_n(x) \xrightarrow[n\to\infty]{} 0$

(ii) The limit function is

$$f(x) = \begin{cases} 0 & x \in [0,\pi] \setminus \{\frac{\pi}{2}\} \\ 1 & x = \frac{\pi}{2} \end{cases}$$

So f is not continuous, even though every f_n is.

(iii) The uniform limit of continuous functions on a compact interval is continuous. Since here the pointwise limit is a discontinuous function, the limit cannot be uniform.

<u>Alternative answer:</u> Fix $\varepsilon = \frac{1}{2}$.
Since $f_n(x)$ is continuous with $f_n(0) = 0$ and $f_n(\frac{\pi}{2}) = 1$, for every n there exists an $x \in [0, \frac{\pi}{2}]$ s.t. $f_n(x) = \frac{2}{3}$.

But then

$$|f_n(x) - f(x)| = \left|\frac{2}{3} - 0\right| > \frac{1}{2} = \varepsilon.$$

This proves that f_n does not converge uniformly to f.

(iv) Theorem 5.6.8 says that $\left(C([a,b];\mathbb{F}), \|\cdot\|_{L^\infty} \right)$ is a Banach space, hence complete.

In other words, sequences that converge in the L^∞-norm have a continuous limit. But this notion of convergence is equivalent to uniform convergence, which is not the case in this example.

5.42 (i) We need to check that

$$\sum_{k=0}^{\infty} \left\| (-1)^k \frac{A^{2k}}{(2k)!} \right\| \le \sum_{k=0}^{\infty} \underbrace{\frac{\|A\|^{2k}}{(2k)!}}_{=: c_k}$$

converges. But

$$\frac{c_{k+1}}{c_k} = \frac{\|A\|^{2(k+1)}}{(2(k+1))!} \frac{(2k)!}{\|A\|^{2k}}$$

$$= \frac{\|A\|^2}{(2k+1)(2k+2)} \xrightarrow[k\to\infty]{} 0$$

So by the ratio test, $\sum_{k=0}^{\infty} c_k$ converges

so that $\sum_{k=0}^{\infty} (-1)^k \frac{A^{2k}}{(2k)!}$ converges absolutely.