Solutions - Homework Week 9

5.39 $$f_n(x) = \frac{nx}{nx+1} \quad \text{on } [0,1]$$

If $x=0$, $f_n(x) = 0 \longrightarrow 0$ as $n\to\infty$,

if $x \in (0,1]$, $f_n(x) = \frac{x}{x+\frac{1}{n}} \longrightarrow \frac{x}{x} = 1$ as $n \to \infty$.

Hence, the pointwise limit is discontinuous, so the limit cannot be uniform

5.40 $$f_n(x) = \frac{x}{1+x^n} \quad \text{on } [0,1]$$

For $x=1$, $f_n(x) = \frac{1}{1+1^n} = \frac{1}{2} \xrightarrow[n\to\infty]{} \frac{1}{2}$,

for $x \in [0,1)$, $f_n(x) \xrightarrow[n\to\infty]{} \frac{x}{1} = x$

Again, the pointwise limit is a discontinuous function, so the limit cannot be uniform.

5.42 (ii) $$\sum_{k=0}^{\infty} \left\| (-1)^k \frac{A^{2k+1}}{(2k+1)!} \right\| \leq \sum_{k=0}^{\infty} \underbrace{\frac{\|A\|^{2k+1}}{(2k+1)!}}_{=:\, C_k}$$

$$\frac{C_{k+1}}{C_k} = \frac{\|A\|^{2k+3}}{(2k+3)!} \, \frac{(2k+1)!}{\|A\|^{2k+1}} = \frac{\|A\|^2}{(2k+2)(2k+3)} \xrightarrow[k\to\infty]{} 0$$

Thus, $\sum_{k=0}^{\infty} C_k$ converges, hence $\sin(A)$ converges absolutely.

(iii) $$\sum_{k=0}^{\infty} \left\| (-1)^{k-1} \frac{A^k}{k} \right\| \leq \sum_{k=0}^{\infty} \underbrace{\frac{\|A\|^k}{k}}_{=:\, c_k}$$

$$\frac{c_{k+1}}{c_k} = \frac{\|A\|^{k+1}}{k+1} \frac{k}{\|A\|^k} = \frac{k}{k+1} \|A\| \xrightarrow[k\to\infty]{} \|A\|$$

Thus, by the ratio test, $\sum_{k=0}^{\infty} c_k$ converges provided $\|A\|<1$, so in this case,

$$\log(I+A) = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{A^k}{k}$$

converges absolutely.

5.43 $A \in B(X)$ with $\|A\| < 1$. Then

(i) $I-A$ is invertible with $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$,

(ii) $\|(I-A)^{-1}\| \leq \dfrac{1}{1-\|A\|}$.

<u>Proof</u>: (i) Let us test whether

$$\left(\sum_{k=0}^{\infty} A^k\right)(I-A) = (I-A)\sum_{k=0}^{\infty} A^k = I.$$

<u>Note</u>: X could be infinite dimensional, so the left inverse may not be the right inverse!

We compute

$$\left(\sum_{k=0}^{n} A^k\right)(I-A) = A^0 + A + \dots + A^n - (A + A^2 + \dots + A^{n+1})$$

$$= I - A^{n+1}$$

$$\Rightarrow \quad I - \sum_{k=0}^{n} A^k (I-A) = A^{n+1}$$

$$\Rightarrow \quad \left\| I - \sum_{k=0}^{n} A^k (I-A) \right\| \leq \|A\|^{n+1} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow \quad I = \sum_{k=0}^{\infty} A^k (I-A)$$

A similar computation shows that

$$I = (I-A) \sum_{k=0}^{\infty} A^k ,$$

so $\sum_{k=0}^{\infty} A^k$ is the left and right inverse of $I-A$.

$\Rightarrow$ $I-A$ is invertible.

(ii) $$\|(I-A)^{-1}\| = \left\| \sum_{k=0}^{\infty} A^k \right\| \leq \sum_{k=0}^{\infty} \|A\|^k \overset{(*)}{=} \frac{1}{1-\|A\|}$$

where the final step is based on the geometric series for real numbers, where

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} .$$