Week 8 – in-class discussion – Solutions

5.13: Take (x,y) with $x^2+y^2 \le 1$.

Let $\varepsilon > 0$ arbitrary. Then $\left((1-\frac{\varepsilon}{2})x, (1-\frac{\varepsilon}{2})y\right) \in B((x,y),\varepsilon)$

and $\left((1-\frac{\varepsilon}{2})x, (1-\frac{\varepsilon}{2})y\right) \in B(0,1)$.

Thus, any $B((x,y),\varepsilon)$ intersects $B(0,1)$, so (x,y) is a limit point of $B(0,1)$.

Vice versa, take (x,y) with $x^2+y^2 =: r > 1$.

Let $(\tilde{x},\tilde{y}) \in B\left((x,y), \frac{r-1}{2}\right)$. Then

$$\tilde{x}^2+\tilde{y}^2 \geq \left(r - \sqrt{|x-\tilde{x}|^2+(y-\tilde{y})^2}\right)^2 \geq \left(\frac{r+1}{2}\right)^2 > 1$$

$\Rightarrow (\tilde{x},\tilde{y}) \notin B(0,1)$, i.e., there is an open ball about (x,y) that does not intersect $B(0,1)$; hence, (x,y) is not a limit point of $B(0,1)$.

5.18: Let $X = [0,1] \cup [1,2]$ with $d(x,y) = |x-y|$

$S := [0,1]$

S is open in X because $B(x,\frac{1}{2}) \subset X$ is fully contained in S.

It is also closed because $S^c = [1,2]$ is open.

<u>Note:</u> S is <u>not</u> open in $\mathbb{R}$; likewise, S^c is <u>not</u> open in $\mathbb{R}$.

$T := [\frac{1}{2}, 1)$

- T is not closed because $1 \in X$ is a limit point of T not contained in T.
- T is not open because every open neighborhood of $\frac{1}{2}$ contains some numbers from $[1, \frac{1}{2})$ which are not in T.

5.19: Let x be a limit point of $N(T)$.

Then for every $k \in \mathbb{N}$, there exists

$$x_k \in N(T) \cap B(x, \tfrac{1}{k})$$

so that

$$x_k \to x \quad \text{as } k \to \infty.$$

Since T is bounded, it is continuous, and

$$\underbrace{\lim_{k \to \infty} \underbrace{T(x_k)}_{=0}}_{=0} = T\Big(\underbrace{\lim_{k \to \infty} x_k}_{=x}\Big)$$

So $T(x) = 0$

$\Rightarrow x \in N(T)$

Thus, $N(T)$ is closed as it contains all of its limit points.

5.23: (i) $f(x) = x^3$ is uniformly continuous on $(0,1)$, but not on $(0,\infty)$:

$$|x^3 - y^3| = |x-y| \cdot |x^2 + xy + y^2| \qquad (*)$$
$$\leq 3|x-y| \qquad \text{on } (0,1)$$

On the other hand, for fixed $|x-y|$, the right hand side of $(*)$ can be made arbitrarily large by making x, y large on $(0,\infty)$. Thus, continuity is not uniform on $(0,1)$

(ii) $f(x) = \frac{\sin x}{x}$ is uniformly continuous on $(0,\infty)$ (thus, also on $(0,1)$):

Note that, in general, if $\sup_{x \in I} |f'(x)| =: M < \infty$,

then $$|f(x) - f(y)| \leq \left| \int_x^y f'(z)\, dz \right| \leq M \left| \int_x^y dz \right| = M|x-y|,$$

so f is uniformly continuous on I.

Here, $$f'(x) = \frac{x \cos x - \sin x}{x^2} \to 0 \quad \text{as } x \to \infty.$$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{2x} = 0$$

by L'Hôpital's rule.

Since f' is continuous on $(0,\infty)$, it assumes a finite minimum and maximum on $(0,\infty)$.

Thus, by the argument above, f is unif. cont. on $(0,\infty)$.

(iii) $f(x) = x \ln x$

Setting $f(0) = \lim_{x \searrow 0} f(x) = 0$, we can extend f continuously onto $[0,1]$. On this closed interval, f is uniformly continuous because it is continuous. So f is uniformly continuous on $(0,1)$.

<u>Note:</u> The argument from (ii) does not work here, because $\lim_{x \to 0} f'(x)$ does not exist.

f is not uniformly continuous on $(0,\infty)$, because, for $x > y$:

$$
\begin{aligned}
f(x) - f(y) &= x\ln x - x\ln y + x\ln y - y\ln y \\
&= x \ln \frac{x}{y} + (x-y)\ln y \\
&\geq (x-y)\ln y
\end{aligned}
$$

For fixed $x-y$, this can be made arbitrarily large by choosing y large.