Week 8 Homework - Solutions

5.5 Case $d(x,y) \geq d(y,z)$:

$$d(x,y) \leq d(x,z) + d(y,z) \qquad \text{(triangle inequality)}$$

$$\Rightarrow \quad d(x,z) \geq d(x,y) - d(y,z) = |d(x,y) - d(y,z)|$$

Case $d(x,y) < d(y,z)$

$$d(y,z) \leq d(x,y) + d(x,z) \qquad \text{(triangle inequality)}$$

$$\Rightarrow d(x,z) \geq d(y,z) - d(x,y) = |d(x,y) - d(y,z)|$$

5.12

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = 0 \\ \dfrac{2x^2y}{x^4+y^2} & \text{otherwise} \end{cases}$$

(i) $$\lim_{t\to 0} f(t, at) = \lim_{t \to 0} \frac{2at^3}{t^4 + a^2t^2} = 0$$

(ii) $$\lim_{t\to 0} f(t, t^2) = \lim_{t\to 0} \frac{2t^4}{t^4 + (t^2)^2} = 1$$

$\Rightarrow$ f is not continuous at zero because the path-wise limits along different paths do not coincide.

5.17 "$\Rightarrow$": Suppose $x_0 \in E \Rightarrow \inf_{x \in E} d(x, x_0) = 0$

Suppose x_0 is a limit point of E

$\Rightarrow \forall k \in \mathbb{N} \ \exists x_k \in E \cap B(x_0, \frac{1}{k})$

$\Rightarrow \inf_{x \in E} d(x, x_0) \leq d(x_k, x_0) \leq \frac{1}{k} \to 0$ as $k \to \infty$.

$\Rightarrow \inf_{x \in E} d(x, x_0) = 0$ in all cases.

"$\Leftarrow$": $\inf_{x \in E} d(x, x_0) = 0$

$\Rightarrow \forall B(x_0, \varepsilon) \ \exists x \in E$ s.t. $x \in B(x_0, \varepsilon)$

$\Rightarrow$ Either $x_0 \in E$ or x_0 is a limit point of E.

5.20: (i) $f(x,y) = \dfrac{\sqrt{|xy|}}{x^2+y^2}$ (Use $|\cdot|$ under the root to ensure f is defined on $\mathbb{R}^2 \setminus \{0\}$)

$f(x,x) = \dfrac{|x|}{2x^2} \longrightarrow \infty$ as $x \to 0$

so limit does not exist.

(ii) $f(x,y) = \dfrac{xy}{x^2+y^2}$

$f(x,x) = \dfrac{x^2}{x^2+x^2} = \frac{1}{2}$, $f(x,-x) = \dfrac{-x^2}{2x^2} = -\frac{1}{2}$

$\Rightarrow$ limit as $(x,y) \to 0$ does not exist.

(iii) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

$$|f(x,y)| = \frac{|x|\,|y|}{\sqrt{x^2+y^2}}$$

$$\leq \frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} \to 0 \quad \text{as} \quad (x,y) \to 0$$

5.26: (i) Let $B \subset \mathbb{R}^n$ be bounded

Suppose $f(B)$ not bounded.

$\Rightarrow \forall k \in \mathbb{N} \; \exists x_k \in B$ s.t. $|f(x_k)| \geq k$ $\qquad (*)$

Since $B \subset \mathbb{R}^n$ bounded, $\overline{B}$ is compact

$\Rightarrow$ There exists a subsequence of $(x_k)_{k\in\mathbb{N}}$ that converges in $\overline{B}$, hence is Cauchy in B

Since f is uniformly continuous, $(f(x_k))_{k\in\mathbb{N}}$ has a subsequence that is Cauchy.

This contradicts $(*)$.

(ii) Take $f(x) = \dfrac{1}{1-x^2}$ on $(0,1)$, where $f((0,1)) = (1,\infty)$.