# Solutions – Homework Week 7

4.29

$$A^H A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is clearly a rank-1 matrix, so it can have only one non-zero eigenvector. By direct inspection,

$$A^H A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 15 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad \sigma_1^2 = 15$

So the reduced SVD has matrices $\Sigma_1 = (\sqrt{15})$,

$$V_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$$U_1 = A V_1 \Sigma_1^{-1} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{15}} \cdot \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad U_1 = \frac{1}{3}\frac{1}{\sqrt{5}}\begin{pmatrix}3\\0\\6\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}$$

Thus, the reduced SVD reads

$$A = U_1 \Sigma_1 V_1^H = \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}}_{U_1} \cdot \underbrace{\sqrt{15}}_{\Sigma_1} \cdot \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1 & 0\end{pmatrix}}_{V_1^H}$$

To get a full SVD, we need to complete the columns of $U$ and $V$ to a full ONB of the correct dimension, which is easy to do by inspection:

- The vectors $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\1\end{pmatrix}$ are mutually orthogonal and orthogonal to $U_1$, so after normalizing,

$$U = \begin{pmatrix} \underbrace{\begin{matrix}\frac{1}{\sqrt{5}}\\ 0 \\ \frac{2}{\sqrt{5}}\end{matrix}}_{\text{spans } R(A)} & \underbrace{\begin{matrix}0 & -\frac{2}{\sqrt{5}}\\ 1 & 0 \\ 0 & \frac{1}{\sqrt{5}}\end{matrix}}_{\text{spans } R(A)^\perp = N(A^H)} \end{pmatrix}$$

- Similarly, we complete

$$V = \begin{pmatrix} \underbrace{\begin{matrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ 0\end{matrix}}_{\text{spans } N(A)^\perp = R(A^H)} & \underbrace{\begin{matrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0\\ 0 & -\frac{2}{\sqrt{6}} & 0\\ 0 & 0 & 1\end{matrix}}_{\text{spans } N(A)} \end{pmatrix}$$

(Use Gram-Schmidt if necessary to find upper left 3x3 block.)

Then $\Sigma = \begin{pmatrix} \sqrt{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so that

$$A = U \Sigma V^H .$$

4.32 $\|M\|_F^2 = \operatorname{tr}(M^H M)$

(i) $\|UAV\|_F^2 = \operatorname{tr}(V^H A^H \underbrace{U^H U}_{=I} A V)$

$= \operatorname{tr}(V^H A^H A V)$

$= \operatorname{tr}(\underbrace{V V^H}_{=I} A^H A)$ as $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$= \operatorname{tr}(A^H A) = \|A\|_F^2$

(ii) Write $A = U \Sigma V^H$ as a singular value decomposition

$\Rightarrow \|A\|_F^2 = \|\Sigma\|_F^2$ by (i)

$= \operatorname{tr} \Sigma^H \Sigma = \sigma_1^2 + \dots + \sigma_r^2 .$

4.36 $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has $\det A = 1$, and eigenvalue $\lambda = 1$ with algebraic multiplicity 2:

$$p_A(z) = \det(zI - A) = (z-1)^2 .$$

$$A^H A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

with characteristic polynomial

$$p_{A^HA}(z) = \det\left(zI - A^H A\right) = (z-1)(z-2) - 1$$

$$= z^2 - z - 2z + 2 - 1 = z^2 - 3z + 1$$

$\Rightarrow$ eigenvalues of $A^H A$ are

$$\sigma_{1,2}^2 = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{4}{4}} = \frac{3 \pm \sqrt{5}}{2}$$

So the singular values $\sigma_1$ and $\sigma_2$ are different from $\lambda$.

5.2: The railway lines in France are, with some exceptions, straight lines connecting Paris to the cities along those rays. Thus, to travel between cities not on the same line, one has to go to Paris, change trains (and sometimes stations) before traveling on a different line to the destination.

This is reflected in $d_{SNCF}$ where all points not on the same ray have a distance equal to the sum of their Euclidean distance to the origin (= Paris).

To show that $d_{SNCF}$ is a metric:

(i) Clearly, $d_{SNCF}(x,y) \geq 0$.

If $x=y$, $d_{SNCF} = d_2(x,y) = d_2(x,x) = 0$

If $x \neq y$, either $d_{SNCF} = d_2(x,y)$ or

$d_{SNCF} \geq \max(d_2(x,0), d_2(y,0)) > 0$,

so in both cases, it is strictly positive

(ii) Symmetry is obvious.

(iii) For the triangle inequality, need to distinguish cases:

1. $x,y,z$ on the same ray: $d_{SNCF} = d_2$, so triangle inequality holds.

2. $x,y$ on the same ray, $z$ on a different ray:

$$d_{SNCF}(x,z) = \underbrace{d_2(x,0)} + d_2(z,0)$$

$$\leq d_2(x,y) + d_2(y,0)$$

$$= d_{SNCF}(x,y) + d_{SNCF}(y,z)$$

3. $x,z$ on the same ray, $y$ on a different ray:

$$d_{SNCF}(x,z) = d_2(x,z)$$
$$\leq d_2(x,0) + d_2(z,0)$$
$$\leq d_2(x,0) + d_2(y,0) + d_2(y,0) + d_2(z,0)$$
$$= d_{SNCF}(x,y) + d_{SNCF}(y,z)$$

4. $x, y, z$ on different rays:

$$d_{SNCF}(x, z) = d_2(x, 0) + d_2(z, 0)$$

now continue as in case 3.

$$B(x, \varepsilon) = \{ (r+t)\hat{x} : |t| < \varepsilon \} \quad \text{where } \hat{x} = \frac{x}{\|x\|}$$

provided $\varepsilon \leq \|x\|$.

Otherwise,

$$B(x, \varepsilon) = \{ t\hat{x} : |t| < \|x\| + \varepsilon \} \cup B_2(0, \varepsilon - \|x\|)$$

5.11

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = 0 \\ \dfrac{x^2 - y^2}{\sqrt{x^2+y^2}} & \text{otherwise} \end{cases}$$

We need to show that $\lim_{(x,y)\to 0} f(x,y) = 0 = f(0,0)$

$$|f(x,y) - f(0,0)| \leq \left| \frac{x^2+y^2}{\sqrt{x^2+y^2}} \right| = \|(x,y)\| \longrightarrow 0 \quad \text{as } (x,y) \to 0.$$

Thus, $f$ is continuous at $(x,y) = 0$.