Week 6 – For discussion Thursday, Nov. 20

4.1 Suppose $Ax = \lambda x$, i.e. (x, λ) is an eigenpair

$$\Rightarrow A^k x = A^{k-1}(Ax) = A^{k-1}(\lambda x) = \lambda (A^{k-1} x) = \dots = \lambda^k x$$

Suppose A, in addition, is nilpotent with $A^k = 0$

$$\Rightarrow 0 = A^k x = \lambda^k x$$

Since $x \neq 0$ as an eigenvector, we must have $\lambda^k = 0 \Rightarrow \lambda = 0$

4.2 Take basis $\mathcal{B} = [1, x, x^2]$, so that

$$M_{\mathcal{B}\mathcal{B}}^{D} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} =: M$$

is the matrix representing $D(p) = p'$ w.r.t. $\mathcal{B}$.

So

$$M^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

M is nilpotent, so the only eigenvalue, by 4.1, is 0.

The corresponding eigenvector is $p(x) = 1$, as $p'(x) = 0 = 0 \cdot p(x)$.

4.14 Algebraic multiplicity of $\lambda = 0$: 3 since $p_M(\lambda) = \lambda^3$.

Geometric multiplicity of $\lambda = 0$: 1 (from 4.2).

So M, hence D, is not semi-simple!

4.17 $\quad A$ semisimple $\Rightarrow \quad A = SDS^{-1}$

with $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and S invertible.

$$\Rightarrow \quad p(z) = \det(zI - A)$$
$$= \det(zI - SDS^{-1})$$
$$= \det(S(zI - D)S^{-1})$$
$$= \det(zI - D) = \prod_{i=1}^{n} (z - \lambda_i)$$

(possibly with repeated factors)

$$\Rightarrow \quad p(A) = \prod_{i=1}^{n} (A - \lambda_i I) = \prod_{i=1}^{n} (SDS^{-1} - \lambda_i I)$$

$$= S \underbrace{\left(\prod_{i=1}^{n} D - \lambda_i \right)}_{= \begin{pmatrix} \prod_{i=1}^{n} (\lambda_1 - \lambda_i) & & & 0 \\ & \prod_{i=1}^{n} (\lambda_2 - \lambda_i) & & \\ & & \ddots & \\ 0 & & & \prod_{i=1}^{n} (\lambda_n - \lambda_i) \end{pmatrix} = 0} S^{-1}$$

(Note that each of the products has at least one factor that is zero, so all the products are zero.)

4.22 Given: $ST = TS$ $\quad (*)$

(i) Suppose $x \in T\Sigma_\lambda(S)$.

$\Rightarrow \exists \underbrace{y \in \Sigma_\lambda(S)}_{\Rightarrow Sy = \lambda y}$ s.t. $x = Ty$

$\Rightarrow Sx = STy \overset{(*)}{=} TSy = T(\lambda y) = \lambda Ty = \lambda x$

$\Rightarrow x \in \Sigma_\lambda(S)$

In other words: $T\Sigma_\lambda(S) \subset \Sigma_\lambda(S)$

(ii) S simple $\Rightarrow$ S has n different 1-dimensional eigenspaces, each spanned by some vector $x_i \neq 0$

Since, by (i), $Tx_i = \mu_i x_i$, x_i is an eigenvector of T. Note that the μ_i do not need to be distinct (example: $T=0$), so T need not be simple, but as it has n l.i. eigenvectors, it is semisimple.