Week 6 Homework – Solutions

1. Suppose $A = -A^H$. (i.e. A is skew-Hermitian)

By Schur's lemma, we can write

$$A = UTU^H$$

with T upper triangular and U orthonormal.

$$\Rightarrow A^H = (U^H)^H T^H U^H = UT^HU^H = -A = -UTU^H$$

This implies that $T^H = -T$, so

$$t_{ij} = 0 \quad \text{if } i \neq j$$

$$\overline{t_{ii}} = -t_{ii} \Rightarrow t_{ii} \text{ is purely imaginary}$$

$$\Rightarrow A = UDU^H \quad \text{where } D = \begin{pmatrix} i\omega_1 & & 0 \\ & \ddots & \\ 0 & & i\omega_n \end{pmatrix}, \ \omega_j \in \mathbb{R}.$$

Exercise 4.13:

$$A = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

$$p_A(z) = \det(zI - A) = \left(z - \frac{4}{5}\right)\left(z - \frac{3}{5}\right) - \frac{2}{25}$$

$$= z^2 - \frac{3}{5}z - \frac{4}{5}z + \frac{12}{25} - \frac{2}{25} = z^2 - \frac{7}{5}z + \frac{2}{5}$$

Eigenvalues: $\lambda_{1,2} = \frac{7}{10} \pm \sqrt{\frac{49}{100} - \frac{40}{100}} = \frac{7}{10} \pm \frac{3}{10}$

$$\lambda_1 = 1 \,, \quad \lambda_2 = \frac{2}{5}$$

Eigenspaces: $\left(\underbrace{\lambda_1 I - A}\right) v_1 = 0$

$$= \begin{pmatrix} 1-\frac{4}{5} & -\frac{2}{5} \\ -\frac{1}{5} & 1-\frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

$$\Rightarrow N(\lambda_1 I - A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$\left(\underbrace{\lambda_2 I - A}\right) v_2 = 0$$

$$= \begin{pmatrix} \frac{2}{5}-\frac{4}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{2}{5}-\frac{3}{5} \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{1}{5} \end{pmatrix}$$

$$\Rightarrow N(\lambda_2 I - A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Set $S = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

To compute S^{-1}: $\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right)$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \end{array}\right) \Rightarrow S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

Then $\quad A = SDS^{-1} \quad$ with $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{5} \end{pmatrix}$

Check:

$$SD = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 2 & \frac{2}{5} \\ 1 & -\frac{2}{5} \end{pmatrix}$$

$$SDS^{-1} = \frac{1}{3}\begin{pmatrix} 2 & \frac{2}{5} \\ 1 & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{12}{5} & \frac{6}{5} \\ \frac{3}{5} & \frac{9}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

$$= A$$

4.16: $\quad A^n = SD^nS^{-1}$

(i) We conjecture: $\quad \bar{D} := \begin{pmatrix} \lim_{n\to\infty} \lambda_1^n & 0 \\ 0 & \lim_{n\to\infty} \lambda_2^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

so $\quad B := S\bar{D}S^{-1}$

$$= \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

Rigorous verification as requested in the problem statement:

$$A^k - B = S(D^k - \bar{D})S^{-1} = S\begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{5}\right)^k \end{pmatrix}S^{-1}$$

$$\Rightarrow \|A^k - B\|_1 \leq \|S\|_1 \left\| \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{5}\right)^k \end{pmatrix} \right\|_1 \|S^{-1}\|_1$$

where $\left\| \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{5}\right)^k \end{pmatrix} \right\|_1 = \left(\frac{2}{5}\right)^k$ by Theorem 3.5.20

$$\Rightarrow \|A^k - B\|_1 \leq \underbrace{\|S\|_1 \, \|S^{-1}\|_1}_{= \text{const}} \left(\frac{2}{5}\right)^k \to 0 \quad \text{as } k \to \infty. \qquad (*)$$

(ii) For each of the norms, $\left\| \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{5}\right)^k \end{pmatrix} \right\| = \left(\frac{2}{5}\right)^k$.

Thus, the limiting argument remains the same, only the constant in (*) may change, which does not affect the conclusion.

(iii) $$3I + 5A + A^3 = S\underbrace{\left(3I + 5D + D^3\right)}S^{-1}$$

$$= \begin{pmatrix} 3+5+1 & 0 \\ 0 & 3+2+\frac{8}{125} \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & \frac{633}{125} \end{pmatrix}$$

$$= \begin{pmatrix} 18 & \frac{633}{125} \\ 9 & -\frac{633}{125} \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \frac{2883}{125} & \frac{984}{125} \\ \frac{492}{125} & \frac{2391}{125} \end{pmatrix}$$

$$= \frac{1}{375} \begin{pmatrix} 2883 & 984 \\ 492 & 2391 \end{pmatrix}$$

$$= \begin{pmatrix} 7.688 & 2.624 \\ 1.312 & 6.376 \end{pmatrix}$$

4.25: (i) $x_1, \dots, x_n$ is an ONB

$$\underbrace{\left(x_1 x_1^H + \dots + x_n x_n^H\right)}_{A} x_j = x_1 \delta_{1j} + \dots + x_n \delta_{nj} = x_j$$

Since A maps every basis vector onto itself, $A = I$.

(ii) In the general case, $A x_j = \lambda_j x_j$, $j = 1, \dots, n$.

Set $U = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Then $AU = UD \Rightarrow A = UDU^H$

$$= \lambda_1 u_1 u_1^H + \dots + \lambda_n u_n u_n^H.$$

4.27: A pos. def. $\Rightarrow \langle e_j, Ae_j \rangle > 0$ for $j = 1, \dots, n$

But $\langle e_j, A e_j \rangle = e_j^H A e_j = a_{jj}$

$\Rightarrow$ The diagonal elements are real and positive.