Solutions Week 5 - In class

3.16 (i) Suppose $A = QR$ with Q orthonormal, R upper triangular.

$$\Rightarrow \quad A = QDD^{-1}R$$

When $D = \mathrm{diag}(d_1, \dots, d_n)$ is diagonal with $|d_1| = \dots = |d_n| = 1$,

then

$$QD = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} = \begin{pmatrix} | & & | \\ d_1 q_1 & \cdots & d_n q_n \\ | & & | \end{pmatrix},$$

so $\tilde{Q} = QD$ is also orthonormal

$$D^{-1}R = \begin{pmatrix} d_1^{-1} & & 0 \\ & \ddots & \\ 0 & & d_n^{-1} \end{pmatrix} \begin{pmatrix} — & r_1^T & — \\ & \vdots & \\ — & r_n^T & — \end{pmatrix}$$

$$= \begin{pmatrix} — & d_1^{-1} r_1^T & — \\ & \vdots & \\ — & d_n^{-1} r_n^T & — \end{pmatrix},$$

so $\tilde{R} = D^{-1}R$ is also upper triangular.

$$\Rightarrow \quad A = \tilde{Q}\tilde{R} \text{ is another QR-decomposition.}$$

(ii) A invertible $\Rightarrow$ R invertible $\Rightarrow$ $r_{11} \neq 0, \dots, r_{nn} \neq 0$.

So we can set $d_i^{-1} = \dfrac{\overline{r_i}}{|r_i|}$

so that $\tilde{r}_{ii} = \dfrac{\overline{r_i}\, r_i}{|r_i|} = |r_i| > 0$.

3.26 (i) $$\|x\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2$$

$$= \underbrace{\sum_{i=1}^{n} |x_i|^2}_{=\|x\|_2^2} + \underbrace{\sum_{i\neq j} |x_i|\,|x_j|}_{\geq 0}$$

$$\Rightarrow \quad \|x\|_2 \leq \|x_1\|_1$$

$$\|x\|_1 = \langle (1,\dots,1), |x| \rangle \qquad \text{where } |x| = \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \end{pmatrix}$$

$$\leq \underbrace{\|(1,\dots,1)\|_2}_{=\left(\sum_{i=1}^{n} 1^2\right)^{\frac{1}{2}} = \sqrt{n}} \|x\|_2$$

So, altogether, $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$

(ii) $$\|x\|_\infty^2 = \sup_i |x_i|^2 \leq \sum_{i=1}^{n} |x_i|^2 = \|x\|_2^2$$

$$\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2 \leq \sum_{i=1}^{n} \underbrace{\sup_j |x_j|^2}_{=\|x\|_\infty^2} = \|x\|_\infty^2 \underbrace{\sum_{i=1}^{n} 1}_{=n}$$

Altogether: $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$

3.40 Let's write out (iii) only as the other two computations are contained therein.

$$T_A : M_n(\mathbb{F}) \to M_n(\mathbb{F})$$
$$X \mapsto T_A(X) = AX - XA$$

$$\Rightarrow \langle Y, T_A(X)\rangle = \langle Y, AX - XA\rangle$$
$$= tr(\underbrace{Y^H A}_{=(A^H Y)^H} X) - \underbrace{tr(Y^H XA)}_{\overset{(2.25)}{=} tr(AY^H X)}$$
$$= tr((YA^H)^H X)$$

$$= \langle A^H Y, X\rangle - \langle YA^H, X\rangle$$
$$= \langle \underbrace{A^H Y - YA^H}_{= T_{A^H}(Y)}, X\rangle$$

$$\Rightarrow T_A^* = T_{A^H}$$

3.44 If $Ax = b$ does not have a solution, then $b \in R(A)$

$\Rightarrow proj_{R(A)^\perp} b \neq 0 \quad \Rightarrow proj_{N(A^H)} b \neq 0 \quad$ since $R(A)^\perp = N(A^*)$

$\Rightarrow \forall y \in N(A^H)$ s.t. $\langle y, b\rangle \neq 0$.

3.46 (i) Let $x \in \mathcal{N}(A^H A)$ $\Rightarrow$ $A^H A x = 0$ $\Rightarrow$ $Ax \in \mathcal{N}(A^H)$

$Ax \in R(A)$ by definition of the range.

(ii) By the fundamental subspace theorem, $R(A) = \mathcal{N}(A^H)^{\perp}$.

So if $x \in \mathcal{N}(A^H A)$, then $Ax \in \mathcal{N}(A^H)$ and $Ax \perp \mathcal{N}(A^H)$.

This can only be true if $Ax = 0$, i.e. $x \in \mathcal{N}(A)$.

$$\left.\begin{array}{l} \Rightarrow \mathcal{N}(A^H A) \subset \mathcal{N}(A) \\ \text{It is obvious that } \mathcal{N}(A^H A) \supset \mathcal{N}(A) \end{array}\right\} \mathcal{N}(A^H A) = \mathcal{N}(A)$$

(iii) Rank-nullity theorem:

$$n = \operatorname{rank} A + \operatorname{nullity} A = \operatorname{rank} A^H A + \operatorname{nullity} A^H A$$

Since $\mathcal{N}(A^H A) = \mathcal{N}(A)$, the nullities coincide. Then so must the ranks.

(iv) $A \in M_{m \times n}(\mathbb{F})$ has l.i. colums $\Rightarrow$ $\operatorname{rank} A = n$ and $m \geq n$.

$\Rightarrow$ $A^H A \in M_{n \times n}(\mathbb{F})$ and, by (iii), has full rank.

$\Rightarrow$ $A^H A$ is invertible.