Week 5 – Solutions (hand-in problems)

Proof of Theorem 3.5.20, (3.27): The claim is

$$\|A\|_1 = \sup_{j\in 1,\dots,n} \sum_{i=1}^{m} |a_{ij}| .$$

"$\leq$":
$$\|Ax\|_1 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$
$$\leq \sum_{i,j} |a_{ij}|\,|x_j|$$
$$\leq \left(\sup_{j\in 1,\dots,n} \sum_{i=1}^{m} |a_{ij}| \right) \underbrace{\sum_{j=1}^{n} |x_j|}_{=\|x\|_1}$$

$$\Rightarrow \|A\|_1 = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq \sup_{j\in 1,\dots,n} \sum_{i=1}^{m} |a_{ij}| .$$

"$\geq$": Let ℓ be the index where the supremum (here the supremum is actually a maximum) is achieved. Set $x = e_\ell$.

Then
$$\|Ax\|_1 = \|Ae_\ell\| = \sum_{i=1}^{m} |a_{i\ell}|$$
$$= \sup_{j\in 1,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$
$$= \sup_{j\in 1,\dots,n} \sum_{i=1}^{m} |a_{ij}|\,\|x\|_1 \qquad \text{as } \|x\|_1 = 1$$

$$\Rightarrow \|A\|_1 \geq \frac{\|Ax\|_1}{\|x\|_1} = \sup_{j\in 1,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$

3.17: $A \in M_{m \times n}$, $\operatorname{rank} A = n$

$\Rightarrow$ $m \geq n$ and the columns of A are l.i.

Let $A = \hat{Q}\hat{R}$ with $\hat{R} \in M_{n \times n}$ and $\hat{Q} \in M_{m \times n}$ where $\hat{Q}$ has orthonormal columns.

$$A^H A x = A^H b$$

$$\Leftrightarrow \hat{R}^H \hat{Q}^H \hat{Q} \hat{R} x = \hat{R}^H \hat{Q}^H b$$

$$\Leftrightarrow \underbrace{\hat{Q}^H \hat{Q}}_{= I_n \text{ by orthonormality of the columns of } \hat{Q}} \hat{R} x = \hat{Q}^H b$$

because $\operatorname{rank} \hat{R} = n$, so $\hat{R}$ is invertible

$$\Leftrightarrow \hat{R} x = \hat{Q}^H b$$

3.29 $$\|Q\|_2^2 = \sup_{x \neq 0} \frac{\|Qx\|^2}{\|x\|^2} = \sup_{x \neq 0} \frac{\langle Qx, Qx \rangle}{\|x\|^2}$$

$$= \sup_{x \neq 0} \frac{\langle x, x \rangle}{\|x\|^2} = 1$$

Since $R_x(A) = Ax$ is linear also in the parameter x, it suffices to prove that $\|R_x\| = 1$ when $\|x\| = 1$.

So assume that $\|x\| = 1$.

$$\|R_x\| = \sup_{A \neq 0} \frac{\|Ax\|}{\|A\|} = \sup_{A \neq 0} \frac{\|Ax\|}{\sup_{y \neq 0} \frac{\|Ay\|}{\|y\|}} \leq \sup_{A \neq 0} \frac{\|Ax\|}{\frac{\|Ax\|}{\|x\|}} = \|x\| = 1$$

Now select $q_2, \dots, q_n$ such that $\{x, q_2, \dots, q_n\}$ is an ONB,

i.e., the matrix

$$Q = \begin{pmatrix} \text{---} & x^H & \text{---} \\ \text{---} & q_2^H & \text{---} \\ & \vdots & \\ \text{---} & q_n^H & \text{---} \end{pmatrix}$$

is orthonormal.

$$\Rightarrow \quad \|R_x\| = \sup_{A \neq 0} \frac{\|Ax\|}{\|A\|} \geq \frac{\|Qx\|}{\|Q\|} = \frac{\|Qe_1\|}{1} = \|e_1\| = 1$$

This proves that $\|R_x\| = 1$ when $\|x\| = 1$,

so $\|R_x\| = \|x\|$ in general.

3.37 Take, as suggested, the inner product on $L^2([-1,1]; \mathbb{R})$.

According to Example 3.3.3, $q_1 = \frac{1}{\sqrt{2}}$, $q_2 = \sqrt{\frac{3}{2}}\,x$, $q_3 = \sqrt{\frac{5}{8}}\,(3x^2 - 1)$

is an ONB. Further (see Remark 3.7.2), if

$$x = \sum_{i=1}^{n} \langle q_i, x \rangle q_i ,$$

then

$$L(x) = L\left(\sum_{i=1}^{n} \langle q_i, x \rangle q_i\right) = \sum_{i=1}^{n} \langle q_i, x \rangle L(q_i)$$

$$= \Big\langle \underbrace{\sum_{i=1}^{n} \overline{L(q_i)}\, q_i}_{=:y} , x \Big\rangle = \langle y, x \rangle$$

Here, $L(p) = p'(1)$

$$\Rightarrow \quad L(q_1) = q_1'(1) = 0$$

$$L(q_2) = q_2'(1) = \sqrt{\frac{3}{2}}$$

$$L(q_3) = \sqrt{\frac{5}{8}}\,(6x)\Big|_{x=1} = \sqrt{\frac{5}{2}} \cdot 3$$

$$\Rightarrow y = 0 \cdot q_1 + \sqrt{\frac{3}{2}}\, q_2 + \sqrt{\frac{5}{2}} \cdot 3 \cdot q_3$$

$$= \frac{3}{2}x + 3 \cdot \sqrt{\frac{5}{2}} \cdot \sqrt{\frac{5}{8}}\,(3x^2 - 1)$$

$$= \frac{3}{2}x + \frac{15}{4}(3x^2 - 1)$$

<u>Alternative solution:</u>

write
$$p(x) = a_0 + a_1 x + a_2 x^2$$
$$y(x) = b_0 + b_1 x + b_2 x^2$$

$$L(p) = p'(1) = a_1 + 2a_2$$

$$\langle y, p \rangle = \int_{-1}^{1} \underbrace{(b_0 + b_1 x + b_2 x^2)(a_0 + a_1 x + a_2 x^2)}\, dx$$

$$= a_0 b_0 + (a_0 b_1 + b_0 a_1)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2$$
$$+ (b_1 a_2 + b_2 a_1)x^3 + a_2 b_2 x^4$$

$$= 2\left[a_0 b_0 + \frac{a_0 b_2 + a_1 b_1 + a_2 b_0}{3} + \frac{a_2 b_2}{5}\right]$$

$$= a_0\left(2b_0 + 2\frac{b_2}{3}\right) + a_1\left(2\frac{b_1}{3}\right) + a_2\left(\frac{2}{3}b_0 + \frac{2}{5}b_2\right)$$

Comparing coefficients, we obtain the linear system

$$2b_0 + \frac{2}{3}b_2 = 0$$

$$\frac{2}{3}b_1 = 1 \qquad \Rightarrow b_1 = \frac{3}{2}$$

$$\frac{2}{3}b_0 + \frac{2}{5}b_2 = 2$$

$$\Rightarrow \begin{pmatrix} 2 & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} b_0 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{5} & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & \frac{4}{45} & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & -\frac{15}{4} \\ 0 & 1 & \frac{45}{4} \end{array}\right)$$

So $b_0 = -\frac{15}{4}$, $b_2 = \frac{45}{4}$

$$\Rightarrow y(x) = -\frac{15}{4} + \frac{3}{2}x + \frac{45}{4}x^2$$

This coincides with the first computation.

3.47 $A \in M_{m \times n}$ has rank n, $P := A(A^H A)^{-1} A^H$, $A^H A$ invertible by 3.46!

(i) $P^2 = A\underbrace{(A^H A)^{-1} A^H A}_{=I}(A^H A)^{-1} A^H = A(A^H A)^{-1} A^H = P$

(ii) $P^H = (A^H)^H \left((A^H A)^{-1}\right)^H A^H = A \underbrace{\left(A^H (A^H)^H\right)^{-1}}_{=(A^H A)^{-1}} A^H = P$

(iii) A has full rank $\Rightarrow \dim R(A^H) = \dim \underbrace{N(A)^\perp}_{=\mathbb{F}^n} = n \Rightarrow R(A^H) = \mathbb{F}^n$.

$(A^H A)^{-1}$ and A have trivial nullspaces $\Rightarrow$ rank $P = n$.