Week 4 – Solutions for in-class discussion

2.50 If A is reduced to REF by a sequence of elementary row operations and D is reduced to REF by another sequence of elementary row operations, then

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} =: M$$

is reduced to row echelon form by applying the first sequence of transformations to the upper division of rows and the second sequence of transformations to the lower division of rows.

The determinants of the necessary elementary matrices are the same, and the diagonal elements which appear in the REF of the combined matrix are exactly those that appear in the separate REFs of A and D.

Thus, all factors contributing to $\det M$ appear exactly once in the separate computations of $\det A$ and $\det D$.

$$\Rightarrow \quad \det M = \det A \, \det D .$$

2.52
$$\begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} AI + 0 & AA^{-1}B + 0 \\ CI + 0 & CA^{-1}B + D - CA^{-1}B \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix} =: M$$

$$\Rightarrow \det M = \underbrace{\det \begin{pmatrix} A & 0 \\ C & I \end{pmatrix}}_{\overset{(2.50)}{=} \det A \det I} \underbrace{\det \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}}_{\overset{(2.50)}{=} \det I \det(D - CA^{-1}B)}$$

$$= \det A \det(D - CA^{-1}B)$$

3.1 (i) $\|x+y\|^2 - \|x-y\|^2 = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle$

$$= \|x\|^2 + \langle y,x\rangle + \langle x,y\rangle + \|y\|^2 - \|x\|^2 + \langle y,x\rangle + \langle x,y\rangle - \|y\|^2$$

$$= 4 \langle x,y\rangle \qquad (\text{inner product space over } \mathbb{R}!)$$

(ii) $\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 + \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$

$$= 2\left(\|x\|^2 + \|y\|^2\right)$$

3.5 Need to compute $\|(x-1)\|^2$ first:

$$\|(x-1)\|^2 = \int_0^1 (x-1)^2\, dx = \int_{-1}^0 u^2\, du = \frac{1}{3} \qquad u = x-1$$

$$\Rightarrow \operatorname{proj}_{\operatorname{span}\{x-1\}} e^x = (x-1) \cdot \frac{1}{3} \cdot \underbrace{\int_0^1 (x-1)e^x\, dx}_{= \underbrace{(x-1)e^x \Big|_0^1}_{=-(-1)} - \underbrace{\int_0^1 e^x\, dx}_{= e^x \big|_0^1 = e-1}}$$

$$= (x-1)\frac{2-e}{3}$$

3.10 (i) "$\Rightarrow$" Suppose Q is an orthonormal matrix

$$\Rightarrow \quad \langle Qe_i, Qe_j \rangle = \langle e_i, e_j \rangle$$

$$\Rightarrow \quad (Qe_i)^H Qe_j = e_i^H e_j = \delta_{ij}$$

$$\Rightarrow \quad e_i^T Q^H Q e_j = \delta_{ij}$$

$$\Rightarrow \quad Q^H Q = I$$

$\Rightarrow$ There must be a right inverse B of Q s.t. $QB = I$

$$\Rightarrow \quad Q^H QB = Q^H I \Rightarrow B = Q^H$$

$$\Rightarrow \quad QQ^H = I$$

"$\Leftarrow$": Suppose $Q^H Q = I$

$$\Rightarrow \quad x^H Q^H Q y = x^H y$$

$$\Rightarrow \quad \langle Qx, Qy \rangle = \langle x, y \rangle$$

$\Rightarrow$ Q represents an orthonormal transformation w.r.t. standard basis and standard inner product

$\Rightarrow$ Q is orthonormal

(ii) Q orthonormal $\Rightarrow \|Qx\|^2 = \langle Qx, Qx \rangle = \langle x, x \rangle = \|x\|^2$

(iii) $Q^{-1} = Q^H$, so by (i), Q is orthonormal

(iv) $$Q^H Q = \begin{pmatrix} - & q_1^H & - \\ & \vdots & \\ - & q_n^H & - \end{pmatrix} \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} = \begin{pmatrix} q_1^H q_1 & \cdots & q_1^H q_n \\ \vdots & & \\ q_n^H q_1 & \cdots & q_n^H q_n \end{pmatrix} = I$$

$\Rightarrow q_i^H q_j = \delta_{ij}$, i.e. $\{q_1, \dots, q_n\}$ orthonormal

(v) $$1 = \det\left(Q^H Q\right) = \det Q^H \det Q$$
$$= \det \overline{Q}^T \det Q$$
$$= \overline{\det Q} \det Q$$
$$= |\det Q|^2$$

$\Rightarrow |\det Q| = 1$

The converse is obviously not true, since, e.g.

$$\det \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = 1$$

but the matrix is not orthonormal.

(vi) Q_1, Q_2 orthonormal $\Rightarrow$ $Q_1 Q_2 (Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H$
$$= Q_1 I Q_1^H$$
$$= I$$

$\Rightarrow$ $Q_1 Q_2$ is orthonormal by (i).