Solutions for Homework Week 4

2.51 We need to first check the identity as stated:

$$\begin{pmatrix} I & 0 \\ -y^H & 1 \end{pmatrix} \underbrace{\begin{pmatrix} I-xy^H & x \\ 0^H & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ y^H & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} I-xy^H+xy^H & x \\ 0^H+y^H & 1 \end{pmatrix} = \begin{pmatrix} I & x \\ y^H & 1 \end{pmatrix}$$

$$= \begin{pmatrix} I & x \\ -y^H+y^H & -y^Hx+1 \end{pmatrix} = \begin{pmatrix} I & x \\ 0^H & 1-y^Hx \end{pmatrix}$$

Now take the determinant on both sides:

$$\underbrace{\det\begin{pmatrix} I & 0 \\ -y^H & 1 \end{pmatrix}}_{=1} \underbrace{\det\begin{pmatrix} I-xy^H & x \\ 0^H & 1 \end{pmatrix}}_{=\det(I-xy^H)} \underbrace{\det\begin{pmatrix} I & 0 \\ y^H & 1 \end{pmatrix}}_{=1} = \underbrace{\det\begin{pmatrix} I & x \\ 0^H & 1-y^Hx \end{pmatrix}}_{=1-y^Hx}$$

where we have used Exercise 2.50 in the last step on all determinants.

$$\Rightarrow \quad \det(I-xy^H) = 1-y^Hx$$

3.2

$$\|x+y\|^2 - \|x-y\|^2 = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle$$

$$= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 - \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle - \|y\|^2$$

$$= 2\langle x,y\rangle + 2\langle y,x\rangle$$

$$= 2\left(\langle x,y\rangle + \overline{\langle x,y\rangle}\right)$$

$$= 4\,\mathrm{Re}\,\langle x,y\rangle$$

$$\|x-iy\|^2 - \|x+iy\|^2 = \|x\|^2 - \langle x, iy\rangle - \langle iy, x\rangle + \|y\|^2 - \|x\|^2 - \langle x, iy\rangle - \langle iy, x\rangle - \|y\|^2$$

$$= -2\langle x, iy\rangle - 2\langle iy, x\rangle$$

$$= -2i\langle x,y\rangle - 2\bar{i}\langle y,x\rangle$$

$$= -2i\left(\underbrace{\langle x,y\rangle - \overline{\langle x,y\rangle}}_{= 2i\,\mathrm{Im}\,\langle x,y\rangle}\right)$$

$$= 4\,\mathrm{Im}\,\langle x,y\rangle$$

In total:

$$\langle x,y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\left(\|x-iy\|^2 - \|x+iy\|^2\right)\right]$$

3.8 (i) $$\langle \cos x, \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\cos^2 x}_{=\frac{1}{2}(1+\cos 2x)}\, dx$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} dx}_{=2\pi} + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2x\, dx}_{=0 \text{ by periodicity}}$$

$$= 1 \quad \checkmark$$

$$\langle \cos x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\cos x \sin x}_{\frac{1}{2}\sin 2x}\, dx = 0$$

$$\langle \cos x, \cos 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \cos 2x\, dx$$

$$= \underbrace{\frac{1}{\pi}\sin x \cos 2x \Big|_{-\pi}^{\pi}}_{=0} - \frac{1}{\pi}\int_{-\pi}^{\pi}\sin x\,(-2\sin 2x)\, dx$$

$$= \underbrace{-\frac{2}{\pi}\cos x \sin 2x \Big|_{-\pi}^{\pi}}_{=0} + \underbrace{\frac{2}{\pi}\int_{-\pi}^{\pi}\cos x\,(2\cos 2x)\, dx}_{=4\langle \cos x, \cos 2x\rangle}$$

$$\Rightarrow \langle \cos x, \cos 2x\rangle = 0$$

The other pairs can be computed similarly.

(ii) $$\langle t,t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{\pi}\,\frac{1}{3}t^3\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$

$$\Rightarrow \|t\| = \sqrt{\frac{2}{3}}\,\pi$$

(iii) A computation similar to the ones in (i) will show that

$$\langle \cos 3t, \cos t\rangle = \langle \cos 3t, \sin t\rangle = \dots = 0,$$

so $\mathrm{proj}_X(\cos 3t) = 0$

(iv) $\langle t, \cos t\rangle = 0$ (by symmetry)

$$\langle t, \sin kt\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin kt\,dt$$

$$= \underbrace{\frac{1}{\pi}\, t\,\frac{-\cos kt}{k}\Big|_{-\pi}^{\pi}} - \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-\cos kt}{k}\,dt}_{=0}$$

$$= \frac{-1}{\pi}\left(\pi\,\frac{\cos k\pi}{k} + \pi\,\frac{\cos(-k\pi)}{2}\right)$$

$$= 2\cdot\frac{(-1)^{k+1}}{k}$$

$$\langle t, \cos 2t\rangle = 0$$

$$\Rightarrow \mathrm{proj}_X(t) = \underbrace{\langle t, \sin t\rangle}_{=2}\sin t + \underbrace{\langle t, \sin 2t\rangle}_{=-1}\sin 2t$$

$$= 2\sin t - \sin 2t$$

3.11 As soon as x_i is linearly dependent on $x_1, \dots, x_{i-1}$,

$$\mathrm{proj}_{\mathrm{span}\{q_1,\dots,q_{i-1}\}} x_i = \mathrm{proj}_{\mathrm{span}\{x_1,\dots,x_{i-1}\}} x_i = x_i \quad ,$$

so the residual vector of this projection is 0 and must be discarded.

3.15 $a_1 = (1,1,1,1)^T$

$r_{11} = \|a_1\| = 2 \qquad q_1 = \frac{a_1}{r_{11}} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

$a_2 = (-1,4,-1,4)^T$

$$p_1 = \mathrm{proj}_{q_1} a_2 = \underbrace{\langle a_2, q_1\rangle}_{\frac{1}{2}(-1+4-1+4) = 3 =: r_{12}} q_1 = \frac{3}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$x_2 = a_2 - p_1 = \frac{1}{2}\begin{pmatrix}-5\\5\\-5\\5\end{pmatrix}$

$r_{22} = \|x_2\| = \sqrt{\frac{100}{4}} = 5$

$q_2 = \frac{x_2}{r_{22}} = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$

Set $Q = \frac{1}{2}\begin{pmatrix}1 & -1\\1 & 1\\1 & -1\\1 & 1\end{pmatrix}$, $\quad R = \begin{pmatrix}2 & 3\\0 & 5\end{pmatrix}$

Check:

$$QR = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ 2 & 8 \\ 2 & -2 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{pmatrix} = A$$

(ii) $$A^H A x = A^H b \Rightarrow (QR)^H QRx = (QR)^H b$$

$$\Rightarrow R^H Q^H Q R x = R^H Q^H b$$

$$\Rightarrow \underbrace{Q^H Q}_{=I} R x = Q^H b \quad \text{since } R^H \text{ invertible}$$

$$\Rightarrow Rx = Q^H b$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 6 \\ 5 \\ 7 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 17 \\ 9 \end{pmatrix}$$

$$\Rightarrow 5x_2 = \frac{9}{2} \quad \Rightarrow x_2 = \frac{9}{10}$$

$$2x_1 + 3x_2 = \frac{17}{2} \quad \Rightarrow 2x_1 = \frac{17}{2} - \frac{27}{10} = \frac{85-27}{10} = \frac{58}{10}$$

$$\Rightarrow x_1 = \frac{29}{10}$$

$$\Rightarrow x = \frac{1}{10}\begin{pmatrix} 29 \\ 9 \end{pmatrix}$$