Solutions Week 3 - For discussion Oct. 30, 2025

2.13 The rank-nullity theorem for a linear map $K: W \to X$ says:

$$\dim W = \dim R(K) + \dim N(K) \qquad (**)$$

Now restrict the map K to the subspace $R(L) \subset W$, where L is a linear map $L: V \to W$.

By construction, $R(KL) = R(K|_{R(L)})$. But for $K|_{R(L)}$, the rank-nullity theorem says

$$\dim R(L) = \dim R(K|_{R(L)}) + \dim (N(K) \cap R(L))$$

$$\Rightarrow \dim R(KL) = \dim R(L) - \dim (N(K) \cap R(L)) \qquad (*)$$

2.14 (i) From $(*)$: $\dim R(KL) \leq \dim R(L)$

Moreover, $R(KL) \subset R(K)$ because if $x \in R(KL)$, $\exists\, v \in V$ s.t. $x = KLv$ $\Rightarrow x = Kw$ with $w = Lv$ $\Rightarrow x \in R(K)$

$\Rightarrow \dim R(KL) \leq \dim R(K)$

So altogether: $\dim R(KL) \leq \min\{\dim R(K), \dim R(L)\}$

(ii) Clearly, $\dim N(K) \geq \dim (N(K) \cap R(L))$

From $(**)$: $\dim R(KL) \geq \dim R(L) - \dim N(K)$

$\overset{(**)}{=} \dim R(L) + \dim R(K) - \dim W$

221 $S = [e^{i\theta}, e^{-i\theta}]$, $T = [\cos\theta, \sin\theta]$

(i) $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$ $(*)$

$$\Rightarrow M_{TS}^{Id} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

(ii) First option: $M_{ST}^{Id} = \left(M_{TS}^{Id}\right)^{-1}$, so need to compute inv.:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ i & -i & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2i & -i & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2i} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2i} \end{array}\right)$$

$$\Rightarrow M_{ST}^{Id} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Second option: From $(*)$, we easily get

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Rightarrow M_{ST}^{Id} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & -\frac{1}{2i} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \quad \text{as before}$$

(iii) By direct computation ...

2.23 $L(x) = Ax = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_k \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$

$= x_1 a_1 + \dots + x_k a_k$ by rules of matrix multiplication.

$\Rightarrow L(x) \in \operatorname{span}\{a_1, \dots, a_k\}$

vice versa, if $y \in \operatorname{span}\{a_1, \dots, a_k\}$, the computation above shows that $y = L(x)$ where the elements of x are the coefficients of the l.c.

2.30 Suppose $\mathbb{F}^m \cong \mathbb{F}^n$. Then there is a bijective map $L: \mathbb{F}^m \to \mathbb{F}^n$. Since L injective, $N(L) = \{0\}$.
Since L surjective, $R(L) = \mathbb{F}^n$.

Then the rank-nullity theorem reads:

$$\underbrace{\dim \mathbb{F}^m}_{=m} = \underbrace{\dim R(L)}_{=n} + \underbrace{\dim N(L)}_{=0}$$

$\Rightarrow m = n$.