Solutions - Week 3, Homework

2.19 Let's first use the basis $T = [1, x, x^2]$.

$$L[1] = 1 + 4\,1' = 1$$

$$L[x] = x + 4x' = 4 + x$$

$$L[x^2] = x^2 + 4(x^2)' = 8x + x^2$$

$$\Rightarrow \quad M^L_{TT} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}$$

By inspection, we see that, with $S = [x^2+1, x-1, 1]$

$$M^{Id}_{TS} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Need to compute the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} \underbrace{\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{array}}_{= M^{Id}_{ST}}\right)$$

$$\Rightarrow \quad M^L_{SS} = M^{Id}_{ST}\, M^L_{TT}\, M^{Id}_{TS} = M^{Id}_{ST} \begin{pmatrix} 1 & 3 & 1 \\ 8 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 4 & 1 \end{pmatrix}$$

Alternative solution:

$$L[x^2+1] = x^2+1 + 4\cdot 2x$$

$$= 8(x-1) + 8 + (x^2+1) - 1 + 1$$

$$= 8s_3 + 8s_2 + s_1$$

$$L[x-1] = x-1 + 4\cdot 1 = (x-1) + 4 = s_2 + 4s_3$$

$$L[1] = 1 + 4\cdot 0 = s_3$$

$$\Rightarrow \quad M^L_{SS} = \begin{pmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 4 & 1 \end{pmatrix}$$

(The alternative solution looks easier, but this is only due to the fact that the expansion of $L[s_i]$ in terms of s_1, s_2, s_3 is particularly easy to spot without much computation.)

2.20 $S = [e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}]$

$$D[s_1] = (e^{\alpha x})' = \alpha e^{\alpha x} = \alpha s_1$$

$$D[s_2] = (xe^{\alpha x})' = e^{\alpha x} + \alpha x e^{\alpha x} = s_1 + \alpha s_2$$

$$D[s_3] = (x^2 e^{\alpha x})' = 2x e^{\alpha x} + \alpha x^2 e^{\alpha x} = 2s_2 + \alpha s_3$$

$$\Rightarrow \quad M^D_{SS} = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 2 \\ 0 & 0 & \alpha \end{pmatrix}$$

2.25 $A = (a_{ij})$ $\qquad B = (b_{ij})$

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii} \qquad (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

(i) $\operatorname{tr} AB = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik} = \operatorname{tr} BA$

(ii) A, B similar $\Rightarrow \exists S$ invertible s.t. $A = SBS^{-1}$

$$\Rightarrow \operatorname{tr} A = \operatorname{tr}(SBS^{-1}) \overset{(i)}{=} \operatorname{tr}(S^{-1}SB) = \operatorname{tr}(IB) = \operatorname{tr} B$$

(iii) $\operatorname{tr} AB^T = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ik} = \operatorname{tr} B^T A$

by direct application of the definition of trace, matrix product, and transpose.

Notes:
- $\langle A, B \rangle = \operatorname{tr}(A^H B)$ defines an inner product ("Frobenius inner product") that is often used in matrix optimization problems.
- (i) implies that $\operatorname{tr}(A_1 \cdots A_k) = \operatorname{tr}(A_2 \cdots A_k A_1)$
$\qquad = \operatorname{tr}(A_3 \cdots A_k A_1 A_2)$
$\qquad = \ldots$

  I.e., the trace is invariant under cyclic permutations of matrix products.
- (i) remains true if $A \in M_{m \times n}(\mathbb{F})$, $B \in M_{n \times m}(\mathbb{F})$

  (iii) remains true if $A, B \in M_{m \times n}(\mathbb{F})$

2.38 (i) $$(e_k)_i = \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}$$ (Standard unit vector)

$$\Rightarrow \left(e_k e_\ell^T\right)_{ij} = \begin{cases} 1 & \text{if } i=k,\ j=\ell \\ 0 & \text{otherwise} \end{cases}$$

(ii) Direct proof – check the property of the inverse:

(Note that, WLOG, we can take $a=1$ if we do not require that u, v are unit vectors.)

$$\left(I - uv^T\right)\left(I - \frac{uv^T}{v^Tu - 1}\right) = I - uv^T - \frac{uv^T}{v^Tu-1} + \frac{uv^Tuv^T}{v^Tu-1}$$

$$= I - uv^T \underbrace{\left[1 + \frac{1}{v^Tu-1} - \frac{v^Tu}{v^Tu-1}\right]}$$

$$= \frac{v^Tu - 1 + 1 - v^Tu}{v^Tu-1} = 0$$

$$= I$$

<u>Remark:</u> This is a special case of the Sherman-Morrison formula

$$\left(A + uv^T\right)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$$

For a derivation, see Advanced Programming class notes from Winter 2024/25, Appendix A4.

2.40 B is row equivalent to A if $B = E_1 \cdots E_k A$ where $E_1, \ldots, E_k$ are elementary matrices.

(i) $A = IA$, so A is row-equivalent to itself.

$$\underset{\substack{\uparrow \\ \text{elementary matrix}}}{I}$$

(ii) If B is row equivalent to A, then

$$A = E_k^{-1} \cdots E_1^{-1} B$$

where E_i^{-1} is also an elementary matrix, so A is row equivalent to B

(iii) If A is row-equivalent to B and B is row equivalent to C, then

$$\left.\begin{array}{l} A = E_1 \cdots E_k B \\ B = E_{k+1} \cdots E_p C \end{array}\right\} \quad A = E_1 \cdots E_p C$$

$\Rightarrow$ A is row-equivalent to C.