Week 2 - Solution to exercises

2.1 (i) $L(x,y) = (y,x)$ is linear, because:

$$L(ax_1 + bx_2, ay_1 + by_2) = (ay_1 + by_2, ax_1 + bx_2)$$
$$= a(y_1, x_1) + b(y_2, x_2)$$
$$= a L(x_1, y_1) + b L(x_2, y_2)$$

alternatively $L(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

and we know that matrix multiplication defines a linear transformation in this way.

$\mathcal{N}(L) = \{0\}$ (because $(y,x)=0$ implies $x=0$ and $y=0$)
$R(L) = \mathbb{R}^2$ (by inspection)

(ii) $L(x,y) = (x,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is linear, cf. (i)

$\mathcal{N}(L) = \text{span}\{e_2\}$
$R(L) = \text{span}\{e_1\}$

(iii) Not linear, as $L(0,0) = (0+1, 0+1) = (1,1) \neq 0$

(iv) Not linear, as $L(ax, ay) = ((ax)^2, (ay)^2) = a^2(x^2, y^2)$

$$\neq a L(x,y) = a(x^2, y^2)$$

unless $a = \pm 1$.

2.2 $V = (0,\infty)$ with $x \oplus y = xy$

$a \odot x = x^a$

$T(x) = \log x$ maps $(0,\infty)$ into $\mathbb{R}$.

$$T(a\odot x \oplus b \odot y) = \log (x^a y^b)$$
$$= a \log x + b \log y$$
$$= a T(x) + b T(y)$$

So T is linear in the sense specified.

2.3 (i) $L[p](x) = x^2$ is not linear, because $L[0](x) = x^2 \neq 0$

(ii) $L[p](x) = x\,p(x)$ is linear, as

$$L[ap+bq](x) = x\,(a p(x) + b q(x))$$
$$= a x p(x) + b x q(x)$$
$$= a L[p](x) + b L[q](x)$$

Note: Setting $S = [1, x, x^2]$ and $T = [1, x, x^2, x^3, x^4]$,

$$M^L_{TS} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$N(L) = \{0\}$ $\quad R(L) = \text{span}\,\{x, x^2, x^3\}$

(iii) $L[p](x) = x^4 + p(x)$

is not linear, because $L[0](x) = x^4 + 0 \neq 0$

(iv) $L[p](x) = (4x^2 - 3x)\, p'(x)$ is linear, as

$$
\begin{aligned}
L[ap+bq](x) &= (4x^2-3x)\,(ap+bq)'(x) \\
&= (4x^2-3x)\,(a\,p'(x) + b\,q'(x)) \\
&= a\,(4x^2-3x)\,p'(x) + b\,(4x^2-3x)\,q'(x) \\
&= a\,L[p](x) + b\,L[q](x)
\end{aligned}
$$

Note: $L[1](x) = 0$

$L[x](x) = (4x^2-3x)\cdot 1 = 4x^2 - 3x$

$L[x^2](x) = (4x^2-3x)(2x) = 8x^3 - 6x^2$

$\Rightarrow$

$$
M^L_{TS} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 4 & -6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}
$$

$N(L) = \operatorname{span}\{1\}$

$R(L) = \operatorname{span}\{4x^2-3x,\ 8x^3-6x^2\}$