Solutions - Week 2 - written homework

1.9 Given: $v \in \text{span}\{v_1, \dots, v_n\} = V$

$\Rightarrow \exists\, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$\Rightarrow 0 = \alpha_0 v + \alpha_1 v_1 + \dots + \alpha_n v_n$ with $\alpha_0 = -1$

$\Rightarrow$ 0 is represented by a nontrivial l.c. of vectors $\{v, v_1, \dots v_n\}$

$\Rightarrow \{v, v_1, \dots, v_n\}$ is l.d.

1.11 Since $\mathbb{F}[x]$ is defined as the set of all linear combinations of elements from $\{1, x, x^2, \dots\}$, we only need to check that they are l.i.. Suppose that

$$a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k = 0 \qquad (*)$$

(i) Set $x = 0 \Rightarrow a_0 = 0$

(ii) Differentiate $(*)$:

$$a_1 + 2a_2 x + \dots + k a_k x^{k-1} = 0$$

(iii) Set $x = 0 \Rightarrow a_1 = 0$

Now repeat argument from (ii). After k steps, we see that

$$a_0 = a_1 = \dots = a_k = 0$$

$\Rightarrow \{1, x, \dots, x^k\}$ is l.i.

Since this is true for any k, we conclude that $\{1, x, x^2, \dots\}$ is l.i.

1.12 $\{v_1, \dots, v_n\}$ l.i. $\Rightarrow$ $n \leq \dim V$ (Theorem from class)

$\Rightarrow$ $\{v_2, \dots, v_n\}$ has strictly less than $\dim V$ elements

$\Rightarrow$ it cannot span V (same Theorem from class, cf. Cor. 1.4.4)

1.14 Given: any vector in V can be uniquely expressed as a l.c. of elements from S,

$\Rightarrow$ $\operatorname{span} S = V$

and S is l.i. (since also the representation of 0 is unique)

$\Rightarrow$ S is basis.

1.23 Given: S is l.i., $|S| = \dim V$

Suppose $\exists$ $v \notin \operatorname{span} S$. Then $\{v\} \cup S$ is l.i. and has $\dim V + 1$ elements. So it cannot be l.i. $\Rightarrow$ contradiction

$\Rightarrow$ S is basis.