Solutions – Week 12

6.31 $\quad g(x,y,z) = e^{2x+yz}$

$$Dg(x,y,z) = \left(2e^{2x+yz},\ ze^{2x+yz},\ ye^{2x+yz}\right)$$

$$\operatorname{Hess} g = \begin{pmatrix} 4e^{2x+yz} & 2ze^{2x+yz} & 2ye^{2x+yz} \\ 2ze^{2x+yz} & z^2e^{2x+yz} & e^{2x+yz} + zye^{2x+yz} \\ 2ye^{2x+yz} & e^{2x+yz} + zye^{2x+yz} & y^2e^{2x+yz} \end{pmatrix}$$

$\Rightarrow \quad g(0,0,0) = 1$

$Dg(0,0,0) = (2, 0, 0)$

$$\operatorname{Hess} g(0,0,0) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad g(x,y,z) = 1 + (2,0,0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2}(x,y,z)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \dots$$

$$= 1 + 2x + 2x^2 + yz + \dots$$

Or: $\quad e^t = 1 + t + \frac{1}{2}t^2 + O(t^3)$

$$\Rightarrow \quad e^{2x+yz} = 1 + (2x+yz) + \frac{1}{2}(2x+yz)^2 + O(\|(x,y,z)\|^3)$$

$$= 1 + 2x + yz + 2x^2 + O(\|(x,y,z)\|^3)$$

6.32 $f(x,y) = \ln(1 + x\sin y)$

$$Df = \left(\frac{\sin y}{1+x\sin y} , \frac{x\cos y}{1+x\sin y} \right)$$

$$D^2 f = \begin{pmatrix} \frac{-(\sin y)^2}{(1+x\sin y)^2} & \frac{\cos y(1+x\sin y) - x\cos y\sin y}{(1+x\sin y)^2} \\ \text{-----} & \frac{-x\sin y(1+x\sin y) - x\cos y(x\cos y)}{(1+x\sin y)^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sin^2 y}{(1+x\sin y)^2} & \frac{\cos y}{(1+x\sin y)^2} \\ \frac{\cos y}{(1+x\sin y)^2} & \frac{-x\sin y - x^2}{(1+x\sin y)^2} \end{pmatrix}$$

$\Rightarrow f(0,0) = 0$

$Df(0,0) = (0,0)$

$$D^2f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow f(x,y) = 0 + (0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(\|(x,y)\|^3)$$

$$= xy + O(\|(x,y)\|^3)$$

Alternatively: $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$

$\sin y = y + O(\|y\|^3)$

$$\Rightarrow f(x,y) = x(y + O(\|y\|^3)) + O(\|(x,y)\|^4) = xy + O(\|(x,y)\|^3)$$

6.34 For simplicity assume $f \in C^3(U, \mathbb{R})$

When $Df(x_0) = 0$, then

$$f(x_0+h) = f(x_0) + \frac{1}{2} h^T D^2 f(x_0) h + \phi(h) \qquad \text{where } \phi(h) = O(\|h\|^3)$$

$$= f(x_0) + \frac{1}{2} \|h\|^2 \left(u^T D^2 f(x_0) u + \psi(h) \right)$$

where $\psi(h) = O(\|h\|)$

$u = \frac{h}{\|h\|}$

Since $D^2 f(x)$ is pos. def. and the set of unit vectors in $\mathbb{R}^n$ is compact, there exist a u such that

$$u^T D^2 f(x_0) u = m > 0$$

is minimal. Since $\psi = O(\|h\|)$, we can take δ small enough such that

$$|\psi(h)| < \frac{m}{2} \quad \text{on } B(x_0, \delta) \; ; \; \delta > 0$$

Then, on $B(x_0, \delta)$, $\quad u^T D^2 f(x_0) u + \psi(h) \geq \frac{m}{2} > 0$,

so that

$$f(x_0+h) > f(x_0) \qquad \text{for all } 0 < \|h\| < \delta .$$

Thus, f has a local maximum at x_0.

Week 12 solutions

Note: 6.28 was done in detail in class.

7.1 (in class discussion): $f(x) = \dfrac{x + \sqrt{x^2+1}}{2}$

To check if this is a contraction, compute

$$|f(x) - f(y)| = \left| \frac{x}{2} + \frac{1}{2}\sqrt{x^2+1} - \frac{y}{2} - \frac{1}{2}\sqrt{y^2+1} \right|$$

$$= \left| \frac{1}{2}(x-y) + \frac{1}{2}\left(\sqrt{x^2+1} - \sqrt{y^2+1}\right) \right|$$

$$= \frac{\left(\sqrt{x^2+1} - \sqrt{y^2+1}\right)\left(\sqrt{x^2+1} + \sqrt{y^2+1}\right)}{\sqrt{x^2+1} + \sqrt{y^2+1}} = \frac{x^2+1-(y^2+1)}{\sqrt{x^2+1} + \sqrt{y^2+1}} = \frac{(x-y)(x+y)}{\sqrt{x^2+1} + \sqrt{y^2+1}}$$

$$= \underbrace{\frac{1}{2}\left(1 + \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}}\right)}_{<1} |x-y| \qquad (\text{note: } x, y \geq 0)$$

< 1 because $x < \sqrt{x^2+1}$, $y < \sqrt{y^2+1}$

However: It is not possible to find λ s.t. $\frac{1}{2}\left(1 + \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}}\right) \leq \lambda < 1$

because $\dfrac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}} \longrightarrow 1$ as $x, y \to \infty$.

Thus, f is NOT a contraction, the CMT does NOT apply.

Note that this is consistent with the observation that f does not have a fixed point.

Indeed: $x = f(x) \Rightarrow x = \frac{x}{2} + \frac{1}{2}\sqrt{x^2+1} \Rightarrow \frac{x}{2} = \frac{1}{2}\sqrt{x^2+1} \Rightarrow x^2 = x^2 + 1$ $\lightning$

7.2: $f(x) = \sqrt{\alpha + x}$

$$\Rightarrow |f(x) - f(y)| = \left|\sqrt{\alpha+x} - \sqrt{\alpha+y}\right| = \left| \frac{\left(\sqrt{\alpha+x} - \sqrt{\alpha+y}\right)\left(\sqrt{\alpha+x} + \sqrt{\alpha+y}\right)}{\sqrt{\alpha+x} + \sqrt{\alpha+y}} \right|$$

$$= \left| \frac{\alpha + x - (\alpha+y)}{\sqrt{\alpha+x} + \sqrt{\alpha+y}} \right| = \frac{1}{\sqrt{\alpha+x} + \sqrt{\alpha+y}} |x-y|$$

$$\leq \frac{1}{2\sqrt{\alpha}} |x-y| \qquad \text{as } x, y \geq 0$$

$\Rightarrow f$ is a contraction with constant $\lambda = \dfrac{1}{2\sqrt{\alpha}} \leq \dfrac{1}{2}$ for $\alpha \geq 1$.

Since f clearly maps the closed set $[0,\infty)$ into itself, the CMT applies.

$\Rightarrow$ f has a unique fixed point $\bar{x} \in [0,\infty)$ and the sequence

$$x_0 = 1$$
$$x_{k+1} = f(x_k)$$

converges to $\bar{x}$. Since

$$\bar{x} = f(\bar{x}) \Rightarrow \bar{x} = \sqrt{\alpha + \bar{x}} \Rightarrow \bar{x}^2 - \bar{x} - \alpha = 0$$

This quadratic equation has solutions $\bar{x} = \frac{1 \pm \sqrt{1+4\alpha}}{2}$, where we have to choose the +-sign since $\bar{x} \in [0,\infty)$.

7.8 (i) Show that $C_b([0,\infty);\mathbb{R})$ with L^∞-norm is Banach.

(This is NOT exam material and was not discussed in detail in class.)

We follow the proof of Theorem 5.6.8:

Step 1: Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $C_b([0,\infty);\mathbb{R})$.

$\Rightarrow$ For fixed $x \in [0,\infty)$, $(f_k(x))_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$

$\Rightarrow$ there exists a pointwise limit $f_k(x) \to f(x)$ as $k\to\infty$.

Step 2: The convergence in Step 1 is uniform.

Let $\varepsilon > 0$.

Since $(f_k)_{k\in\mathbb{N}}$ is Cauchy in the L^∞-norm, $\exists\, N>0$ s.t. $\forall\, m,n \geq N$:

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \qquad \forall\, x \in [0,\infty).$$

Take $m\to\infty$: $\quad |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon \qquad \forall\, x \in [0,\infty)$

Thus, $f_n \to f$ uniformly.

Step 3: Show that f is continuous:

Let $\varepsilon > 0$.

$$|f(x) - f(y)| \leq \underbrace{|f(x) - f_n(x)|}_{<\frac{\varepsilon}{3}\ ①} + \underbrace{|f_n(x) - f_n(y)|}_{<\frac{\varepsilon}{3}\ ②} + \underbrace{|f_n(y) - f(y)|}_{<\frac{\varepsilon}{3}\ ①} < 3\,\frac{\varepsilon}{3} = \varepsilon$$

① Choose n large enough that uniform convergence makes these terms $< \frac{\varepsilon}{3}$ independent of x and y

② Now use continuity of f_n to choose $\delta > 0$ small enough to make this term less than $\frac{\varepsilon}{3}$ for x fixed and $|x-y| < \delta$.

<u>Step 4:</u> We still need to prove that f is bounded.

Let $\varepsilon = 1$.

Since (f_n) is Cauchy in L^∞, $\exists N$ s.t. $|f_n(x) - f_m(x)| < \varepsilon = 1 \quad \forall x \in [0,\infty),\ n,m \geq N$.

$$\Rightarrow \quad |f_n(x)| \leq 1 + |f_m(x)| \leq 1 + \|f_m\|_{L^\infty} =: C_m$$

Now take $n \to \infty$: $\quad |f(x)| \leq C_m \quad \Rightarrow f$ is bounded.

(ii) $$T(x)(t) = \underset{\substack{\uparrow \\ \text{is cont. by} \\ \text{assumption}}}{f(t)} + \frac{\mu}{2} \underbrace{\int_0^t e^{-\mu s} x(s)\, ds}_{\text{is a cont. function of } t \text{ by FTC}}$$

So $T(x)$ is clearly a continuous function of t.

Let's show that it is also bounded:

$$|T(x)(t)| \leq \underbrace{|f(t)|}_{\leq \|f\|_{L^\infty}} + \frac{\mu}{2} \int_0^t e^{-\mu s} \underbrace{|x(s)|}_{\leq \|x\|_{L^\infty}} ds$$

$$\leq \|f\|_{L^\infty} + \underbrace{\frac{\mu}{2} \int_0^t e^{-\mu s}\, ds}_{= \frac{1}{2}\mu \frac{e^{-\mu t} - 1}{-\mu} = \frac{1}{2}(1 - e^{-\mu t}) \leq \frac{1}{2}} \|x\|_{L^\infty} \leq \|f\|_{L^\infty} + \frac{1}{2}\|x\|_{L^\infty}$$

So $T(x)$ is bounded in L^∞ as well.

Thus, T maps into $C_b([0,\infty); \mathbb{R})$.

(iii) (This is exam-relevant!)

$$\|T(x) - T(y)\|_{L^\infty} = \sup_{t \in [0,\infty)} \underbrace{\left| \frac{\mu}{2} \int_0^t e^{-\mu s} x(s)\, ds - \frac{\mu}{2} \int_0^t e^{-\mu s} y(s)\, ds \right|}_{\leq \frac{\mu}{2} \int_0^t e^{-\mu s} |x(s) - y(s)|\, ds} \underset{\substack{\uparrow \\ \text{Computation as in (ii)}}}{\leq} \frac{1}{2}\|x - y\|_{L^\infty}$$

So T is a contraction with constant $\lambda = \frac{1}{2}$.

Since λ does not depend on f, it is also a uniform contraction.

<u>Note:</u> T has the form

$$T(x) = Lx + f$$

where L is the linear (!!!) operator $L : C_b([0,\infty);\mathbb{R}) \to C_b([0,\infty);\mathbb{R})$

Thus: $\|T(x) - T(y)\|_{L^\infty} = \|Lx + f - (Ly + f)\|_{L^\infty} = \|L(x-y)\|_{L^\infty}$

$$\leq \|L\| \; \|x-y\|_{L^\infty}$$

where $\|L\|$ denotes the operator norm. A direct calculation, following the same estimate as in (iii) shows that $\|L\| \leq \frac{1}{2}$.

Note also that L is differentiable with $DL(x)v = Lv$, so that T is differentiable and, since f is a constant,

$$DT(x)v = DL(x)v = Lv \quad .$$

Written explicitly:

$$DT(x)[v](t) = \frac{\mu}{2} \int_0^t e^{-\mu t} \, v(s) \, ds$$

(I prefer putting linear arguments in [ ] and nonlinear arguments in ( ), this is a matter of choice which is done differently in the book.)