Week 11 - Solutions for discussion

6.1: $$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

$$f(0,y) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y}(0,y) = 0$$

$$f(x,0) = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x}(x,0) = 0$$

For $(x,y) \neq (0,0)$:

$$\frac{\partial f}{\partial x} = \frac{3x^2 y(x^6+y^2) - 6x^5\, x^3 y}{(x^6+y^2)^2} = \frac{3x^2y^3 - 3x^8 y}{(x^6+y^2)^2}$$

Along the curve $y = 2x^3$:

$$\frac{\partial f}{\partial x}(x, 2x^3) = \frac{3x^2\, 8x^9 - 6x^8\, x^3}{(x^6+4x^6)^2} = \frac{18}{25}\, \frac{1}{x}$$

$$\longrightarrow \infty \quad \text{as } x \to 0,\ x>0$$

$\Rightarrow \frac{\partial f}{\partial x}$ not continuous at $x=0$. Similarly:

$$\frac{\partial f}{\partial y} = \frac{x^3(x^6+y^2) - 2y\, x^3 y}{(x^6+y^2)^2} = \frac{x^9 - x^3 y^2}{(x^6+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x, 2x^3) = \frac{x^9 - 4x^3 x^6}{(x^6+4x^6)^2} = -\frac{3}{25}\, \frac{1}{x^3} \to \infty \quad \text{as } x \searrow 0.$$

$$\Rightarrow Df = \left(2r \sin r^{-1} + r^2(-r^{-2}) \cos r^{-1}\right) Dr$$

$$= \left(2r \sin r^{-1} - \cos r^{-1}\right) \frac{(x,y)}{r}$$

$$= \left(2 \sin r^{-1} - r^{-1}\cos r^{-1}\right) (x,y) \qquad (*)$$

$$\Rightarrow \|Df\| \leq \left(2\,|\sin r^{-1}| + r^{-1}\,|\cos r^{-1}|\right) r$$

$$\leq 2r + 1$$ which is bounded near $r = 0$.

From $(*)$: $\frac{\partial f}{\partial x} = \left(2 \sin r^{-1} - r^{-1}\cos r^{-1}\right) x$

$$\Rightarrow \frac{\partial f}{\partial x}(x,0) = \underbrace{2x \sin \tfrac{1}{x}}_{\substack{\to 0 \\ \text{as } x \to 0}} - \underbrace{\cos \tfrac{1}{x}}_{\substack{\text{takes all values between} \\ -1 \text{ and } +1 \text{ in any} \\ \text{neighborhood of } 0.}} \qquad \text{for } x > 0$$

$\Rightarrow \frac{\partial f}{\partial x}(x,0)$ is not continuous

$\Rightarrow \frac{\partial f}{\partial x}$ is not continuous

The proof that $\frac{\partial f}{\partial y}$ is not continuous is the same

<u>Claim:</u> $Df(0,0) = (0,0)$

<u>Check:</u> $\left| f(x,y) - \left(\underbrace{f(0,0)}_{=0} + \underbrace{Df(0,0)}_{=0}\binom{x}{y}\right)\right| = \left|r^2 \sin\tfrac{1}{r}\right| \leq r^2 = o(r).$

6.6:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

$$\Rightarrow |f(x,y)| = \frac{|x|\,|y|^2}{r^2} \qquad \text{with } r^2 = x^2+y^2$$

$$\leq \frac{r^3}{r^2} = r \longrightarrow 0 \quad \text{as } r = \|(x,y)\| \to 0.$$

$\Rightarrow f$ is continuous at $(x,y) = 0$.

The partial derivatives exist and are zero as for f in 6.1.

Now suppose f is differentiable. Then

$$f(x,y) = \underbrace{f(0,0)}_{=0} + \underbrace{Df(0,0)}_{=0}\begin{pmatrix} x \\ y \end{pmatrix} + o(\|(x,y)\|)$$

But $\dfrac{f(x,y)}{r} = \dfrac{xy^2}{r^3} \neq o(r)$, so f not diff'able at $(0,0)$.

6.9:

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = 0 \end{cases}$$

Set $r^2 = x^2+y^2 \Rightarrow 2r\,Dr = (2x, 2y) \Rightarrow Dr = \dfrac{(x,y)}{r}$

6.11: (i) $X = C([0,1]; \mathbb{R})$ $\qquad p \in [0,1]$

$e_p(f) := f(p)$

$e_p(af) = (af)(p) = a\, f(p) = a\, e_p(f)$

$e_p(f+g) = (f+g)(p) = f(p) + g(p) = e_p(f) + e_p(g)$

$\Rightarrow$ e_p is linear

$e_p(f+h) = e_p(f) + e_p(h)$

$= e_p(f) + De_p(f)(h)$

if we set $De_p(f) = e_p$

$\Rightarrow$ e_p is differentiable.

(ii) $K : [0,1]^2 \to \mathbb{R}$ cont.

$$\mathcal{K}[f](s) = \int_0^1 K(s,t)\, f(t)\, dt$$

$$\mathcal{K}[af](s) = \int_0^1 K(s,t)\, a f(t)\, dt$$

$$= a \int_0^1 K(s,t)\, f(t)\, dt = a\, \mathcal{K}[f](s)$$

$$\mathcal{K}[f+g](s) = \int_0^1 K(s,t)\,(f(t) + g(t))\, dt$$

$$= \int_0^1 K(s,t)\, f(t)\, dt + \int_0^1 K(s,t)\, g(t)\, dt$$

$$= \mathcal{K}[f](s) + \mathcal{K}[g](s)$$

$\Rightarrow$ $\mathcal{K}$ is a linear operator on X.

(Note that $\mathcal{K}[f]$ is continuous because K is cont., hence $s \mapsto K(s,t)$ is uniformly continuous on the compact set $t \in [0,1]$, so that we can take limits inside of the integral!)

$$\mathcal{K}[f+h](s) = \mathcal{K}[f](s) + \mathcal{K}[h](s)$$

$\Rightarrow$ $\mathcal{K}$ is differentiable with $D\mathcal{K}[f](h) = \mathcal{K}[h]$.

6.12: $X = C([0,1]; \mathbb{R})$

$$T(f) = \int_0^1 t^2 f^2(t)\, dt$$

$$\delta T = \int_0^1 t^2\, 2f(t)\, \delta f(t)\, dt \equiv DT(f)\, \delta f \qquad (*)$$

$$\Rightarrow \quad DT(f) g = 2\int_0^1 t^2 f(t) g(t)\, dt \equiv Lg$$

L is linear by the same computation as for 6.11 (ii), not repeated here.

(*) At this point, this is a directional derivative, we show that it is a Fréchet-derivative later.

To show that L is the Fréchet - derivative, we compute:

$$T(f+h) = \int_0^1 t^2 (f+h)^2(t)\, dt$$

$$= \underbrace{\int_0^1 t^2 f^2(t)\, dt}_{=T(f)} + \underbrace{2\int_0^1 t^2 f(t)\, h(t)\, dt}_{=Lh} + \int_0^1 t^2 h^2(t)\, dt$$

$$\Rightarrow \left| T(f+h) - T(f) - Lh \right| \leq \int_0^1 t^2 \, |h(t)|^2\, dt$$

$$\leq \underbrace{\int_0^1 t^2\, dt}_{=\frac{1}{3}} \;\; \|h\|_{L^\infty}^2 = o(\|h\|_{L^\infty})$$

$\Rightarrow$ T is Fréchet - differentiable with $DT(f)g = Lg$.