Week 11 Exercises – Solutions

6.4 $$f(x,y) = xy\,e^{x+y}$$

$$Df = (\partial_x f, \partial_y f)$$

$$= \left(y e^{x+y} + xy e^{x+y},\ x e^{x+y} + xy e^{x+y}\right)$$

$$= \big(y(1+x),\ x(1+y)\big)\, e^{x+y}$$

$$Df(1,-1) = \big(-1(1+1),\ 1(1-1)\big)\, e^{1-1} = (-2, 0)$$

(i) $D_v f = Df\, v$

At $(x,y) = (1,-1)$: $D_v f = (-2, 0)\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = -\frac{6}{5}$

(ii) $Df(x,y)\, v$ is maximized among all unit vectors v if $Df(x,y)$ aligns with v, i.e. $v = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

<u>Note:</u> For this reason, Df is usually identified with the <u>gradient</u> of f, the vector pointing in the direction of steepest ascent.

6.7

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y} & \text{for } x^2+y \neq 0 \\ 0 & \text{for } x^2+y = 0 \end{cases}$$

$$f(x,0) = f(0,y) = 0 \qquad \text{for all } x,y \in \mathbb{R}$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

But $$f(x,-x^2+x^3) = \frac{x(-x^2+x^3)}{x^2-x^2+x^3} = \frac{-x^3+x^4}{x^3} = -1+x \xrightarrow[x\to 0]{} -1$$

$$f(x,x) = \frac{x^2}{x^2+x} \longrightarrow 0 \quad \text{as } x \to 0$$

$\Rightarrow \lim_{(x,y)\to 0} f(x,y)$ does not exist, so f is not cont. at $(x,y)=(0,0)$.

6.8 $$D_u f(0,0) = \frac{d}{dt} f(u_1 t, u_2 t)\Big|_{t=0} \qquad f(x,y) = \begin{cases} \dfrac{x\,|y|}{\sqrt{x^2+y^2}} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

$$= \frac{d}{dt} \underbrace{\frac{u_1 t\,|u_2 t|}{\sqrt{u_1^2 t^2 + u_2^2 t^2}}}_{= u_1 |u_2| \frac{t\,|t|}{\|u\|\,|t|} = \frac{u_1 |u_2|}{\|u\|} t}\Bigg|_{t=0} = \frac{u_1 |u_2|}{\|u\|} \qquad (*)$$

If f were differentiable at $(0,0)$, then $u \mapsto D_u f(0,0)$ would be a linear transformation, but $(*)$ is clearly not linear.

6.13 $E(t) = e^{At}$

$$E(t+h) = E(t)\underbrace{E(h)}_{\substack{= \sum_{i=0}^{\infty} \frac{(Ah)^i}{i!} \\ = I + Ah + \underbrace{\sum_{i=2}^{\infty} \frac{(Ah)^i}{i!}}_{=:\phi(h)}}}$$

$$\Rightarrow E(t+h) = E(t) + e^{At} A h + e^{At}\phi(h)$$

where $\| e^{At} \phi(h) \| \le \underbrace{\| e^{At} \|}_{\text{const}} h^2 \underbrace{\sum_{i=2}^{\infty} \frac{\|A\|^i}{i!} h^{i-2}}$

$$= \|A\|^2 \sum_{i=0}^{\infty} \frac{\|A\|^i h^i}{(i+2)!}$$

$$\le \underbrace{\|A\|^2 \sum_{i=0}^{\infty} \frac{\|A\|^i}{(i+2)!}}_{\text{const}} \quad \text{for } |h| \le$$

$$= \mathcal{O}(h^2) = o(h)$$

$\Rightarrow E$ is Fréchet-differentiable with

$$DE(t)h = e^{At} A h .$$

6.16 (Proof of Prop. 6.4.6)

(i) $f(x) = u(x)^T v(x)$

$$\Rightarrow \delta f(x) = \underbrace{\left(Du(x)\,\delta x\right)^T v(x)}_{= v(x)^T Du(x)\,\delta x} + u(x)^T Dv(x)\,\delta x$$

$$= \left(v(x)^T Du(x) + u(x)^T Dv(x)\right)\delta x$$

$$\Rightarrow Df(x) = v(x)^T Du(x) + u(x)^T Dv(x)$$

(ii) $g(x) = x^T A x$

$$\delta g(x) = \underbrace{(\delta x)^T A x}_{= x^T A^T \delta x} + x^T A\,\delta x = x^T\left(A^T + A\right)\delta x$$

$$\Rightarrow Dg(x) = x^T\left(A^T + A\right)$$

(iii) $H(x) = B(x)\,w(x)$

$$\delta H(x) = \left(DB(x)\,\delta x\right) w(x) + B(x) Dw(x)\,\delta x$$

now write $B(x) = \begin{pmatrix} - & b_1(x) & - \\ & \vdots & \\ - & b_k(x) & - \end{pmatrix}$

$$\Rightarrow DB(x)\,\delta x = \begin{pmatrix} - & Db_1(x)\,\delta x & - \\ & \vdots & \\ - & \underbrace{Db_k(x)\,\delta x}_{m \text{ columns}} & - \end{pmatrix}$$

so that $\left(DB(x)\,\delta x\right)w = \underbrace{\begin{pmatrix} - w^T DB_1^T(x) - \\ \vdots \\ - w^T DB_k^T(x) - \end{pmatrix}}_{n \text{ columns}} \delta x$

$$\Rightarrow DH(x) = \begin{pmatrix} - w^T DB_1^T(x) - \\ \vdots \\ - w^T DB_k^T(x) - \end{pmatrix} + B(x)\,Dw(x)$$

6.17 $\quad F(\gamma(t)) = C$

$\Rightarrow F(\gamma(t))' = 0$

$\Rightarrow DF(\gamma(t))\,\gamma'(t) = 0 \qquad$ or $\langle DF(\gamma(t)), \gamma'(t)\rangle = 0$

$\Rightarrow DF(\gamma(t)) \perp \gamma'(t)$.

6.18 $\quad f(x,y) = (\sin(y) - x,\ e^x - y)^T \qquad g(x,y) = (xy,\ x^2 + y^2)^T$

$Df = \begin{pmatrix} -1 & \cos y \\ e^x & -1 \end{pmatrix} \qquad Dg(x,y) = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$

$f(0,0) = (0,1)^T$

$D(g \circ f)(0,0) = Dg \circ f(0,0)\, Df(0,0) = Dg(0,1)\, Df(0,0)$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$$