Solutions to exercises

1.1 V is closed under

- $\oplus$ because the product of positive reals is positive
- $\odot$ because any power of a positive number is positive

Let's check the vector space axioms:

(i) $x \oplus y = xy = yx = y + x \qquad \forall x, y \in V = (0, \infty)$

(ii) $(x \oplus y) \oplus z = (xy)z = x(yz) = x \oplus (y \oplus z) \qquad \forall x, y, z \in V$

(iii) The zero vector 0_V is represented by $1_{\mathbb{R}}$:

$$0_V \oplus x = 1_{\mathbb{R}} x = x$$

(iv) The additive inverse $\ominus x$ is represented by $\frac{1}{x} \in (0, \infty)$:

$$x \oplus (\ominus x) = x \frac{1}{x} = 1_{\mathbb{R}} = 0_V$$

(v) $a \odot (x \oplus y) = (xy)^a = x^a y^a = a \odot x \oplus a \odot y$

(vi) $(a + b) \odot x = x^{a+b} = x^a x^b = a \odot x \oplus b \odot x$

(vii) $1 \odot x = x^1 = x$

(viii) $(ab) \odot x = x^{ab} = (x^b)^a = a \odot (b \odot x)$

1.3 (i) $(0,0,0) \notin \{(x_1, x_2, x_3) : 3x_1 + 4x_3 = 1\}$,

so this set cannot be a subspace.

(ii) $W = \{(x_1, x_2, x_3) : x_1 = 2x_2 = x_3\}$

Let $x = (x_1, x_2, x_3) \in W$

$y = (y_1, y_2, y_3) \in W$

Set $z = x + y$

$\Rightarrow \quad z_1 = x_1 + y_1 = 2x_2 + 2y_2 = 2(x_2 + y_2) = 2z_2$

$2z_2 = 2(x_2 + y_2) = 2x_2 + 2y_2 = x_3 + y_3 = z_3$

$\Rightarrow z \in W$, i.e. W is closed under addition

Moreover: $ax_1 = a(2x_2) = 2(ax_2)$

$2(ax_2) = a(2x_2) = ax_3$

$\Rightarrow ax \in W$, i.e. W is closed under scalar multiplication

$\Rightarrow W$ is a subspace of $\mathbb{F}^3$.

Note: W is a line through the origin, so is a subspace.

(iii) $W = \{(x_1, x_2, x_3) : x_3 = x_1 + 4x_2\}$ is a plane containing the origin, so is also a subspace.

(An explicit proof can be done as in (ii), but is boring...)

(iv) $(1,1,2) \in \{(x_1,x_2,x_3) : x_3 = 2x_1\}$

$(0,1,3) \in \{(x_1,x_2,x_3) : x_3 = 3x_2\}$

$(1,1,2) + (0,1,3) = (1,2,5) \notin \{(x_1,x_2,x_3) : x_3 = 2x_1 \text{ or } x_3 = 3x_2\}$

Or: the union of two non-identical planes is never a vector space!

1.4 (i) Let $p(x) = x^2 + x$, $q(x) = x^2$.
Both are polynomials of even degree, but $p - q = x$ is not $\Rightarrow$ not closed!

(ii) Not a subspace, as it does not contain 0.

(iii) $p(0) = 0$ is a condition that persists under addition and scalar multiplication:

$$(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$$
$$(ap)(0) = a\,p(0) = a \cdot 0 = 0$$

(iv) As for (iii):
$$(p+q)'(0) = p'(0) + q'(0) = 0 + 0 = 0$$
$$(ap)'(0) = a\,p'(0) = a \cdot 0 = 0$$

(v) No, e.g. $p(x) = 1 + x$ and $q(x) = -x$ both have at least one real root, but $(p+q)(x) = 1 + x - x = 1$ does not.

$\Rightarrow$ Not closed under addition.

1.5 Suppose $x \in \ell^p$, i.e. $\sum_{i=1}^{\infty} |x_i|^p < \infty$

$\Rightarrow \exists\, n$ s.t. $|x_i| \leq 1 \quad \forall i \geq n$

For those elements, $|x_i|^p \geq |x_i|^q$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i|^q \leq \underbrace{\sum_{i=1}^{n-1} |x_i|^q}_{\text{finite as a finite sum}} + \underbrace{\sum_{i=n}^{\infty} |x_i|^p}_{\text{finite because } x \in \ell^p}$$

1.6 $x_1 = (-1, 2, 3)$, $x_2 = (3, 4, 2)$

(i) For $x = (2, 6, 6)$ to be in span$\{x_1, x_2\}$, need to solve

$$a_1 x_1 + a_2 x_2 = x$$

which gives a linear system with augmented matrix

$$\left(\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 10 & 10 \\ 0 & 11 & 12 \end{array}\right)$$

$\Rightarrow$ the system is inconsistent, so $x \notin$ span$\{x_1, x_2\}$.

(ii)

$$\left(\begin{array}{cc|c} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 3 & -9 \\ 0 & 10 & -20 \\ 0 & 11 & -22 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow a_1 = 3, \; a_2 = -2 \;;\quad y = (-9, -2, 5) = 3x_1 - 2x_2$

1.7 (i) $\{1, x^2, x^2-3\}$ does not span $\mathbb{F}[x,2]$

because x cannot be written as a l.c. of the given vectors.

(In detail: $x = a_0 \cdot 1 + a_1 \cdot x^2 + a_2 \cdot (x^2-3)$

$$= (a_0 - 3a_2) \cdot 1 + (a_1 + a_2) x^2$$

equating coefficients, this does not have a solution.)

(ii) Yes, in detail: Let a_0, a_1, a_2 be given. We need to solve

$$a_0 + a_1 x + a_2 x^2 = b_0 + b_1 (x-1) + b_2 (x^2-1)$$

$$= (b_0 - b_1 - b_2) + b_1 x + b_2 x^2$$

This clearly has a solution

(iii) Need to solve

$$a_0 + a_1 x + a_2 x^2 = b_0 (x+2) + b_1 (x-2) + b_2 (x^2-2)$$

$$= (2b_0 - 2b_1 - 2b_2) \cdot 1 + (b_0 + b_1) x + b_2 x^2$$

System matrix is

$$\begin{pmatrix} 2 & -2 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is clearly non-singular.

$\Rightarrow$ The given set spans $\mathbb{F}[x,2]$

(iv) Cannot span, too few elements

(v) Cannot span as x^2 is not representable.

1.8 (i) l.d. because it has three elements that don't span

(ii) l.i. (dimension formula)

(iii) same

(iv) l.i. by inspection: one vector is not multiple of other

(v) l.d. (too many vectors to be l.i.!)

1.15 $p_0 = 1 + x^n$, $p_1 = x + x^n$, $p_2 = x^2 + x^n$, $\ldots$, $p_n = x^n$

gives system matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \ddots & 1 \end{pmatrix}$$

which is clearly non-singular, so $p_0, \ldots, p_n$ is a basis.

1.16 Any partition of $\{1, x, x^2, \ldots\}$ into n disjoint subsets gives rise to a direct sum of their spans which contains all of $\mathbb{F}[n]$.