V vector space $\xrightarrow{\quad L \quad}$ W vector space

S basis of V $\qquad\qquad$ T basis of W

Look for a matrix M^L_{TS} so that the mapping L is represented via matrix multiplication.

<u>Fact:</u> $W; T \xrightarrow{\ K\ } X; U$

$$(K \circ L)(x) = K(L(x))$$

then

$$M^{K\circ L}_{US} = M^K_{UT} \cdot M^L_{TS}$$

need to match

<u>Example:</u> Bernstein Polynomials:

$$1 = x + (1-x) = \left(x + (1-x)\right)^n = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}$$

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!} \qquad n! = 1 \cdot 2 \cdot \ldots \cdot n$$

<u>For $n=2$:</u>

$$1 = \left(x + (1-x)\right)^2 = \underbrace{x^2}_{B^2_2} + \underbrace{2x(1-x)}_{B^2_1} + \underbrace{(1-x)^2}_{B^2_0}$$

$$S = [B^2_0, B^2_1, B^2_2] \qquad T = [1, x, x^2]$$

$$L: \mathbb{F}[x,2] \to \mathbb{F}[x,2] \qquad p \mapsto p'$$

$$L(1) = 1' = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 \quad \leftarrow 1^{st} \text{ column}$$

$$L(x) = x' = 1 = 1\cdot 1 + 0\cdot x + 0 \cdot x^2$$

$$L(x^2) = (x^2)' = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$

$$M^L_{TT} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>Goal:</u> Write out $M^L_{SS} = M^{Id}_{ST} \, M^L_{TT} \, M^{Id}_{TS}$

$M^{Id}_{ST} = \left(M^{Id}_{TS}\right)^{-1}$

M^{Id}_{TS} "easy"

$$B^2_0 = (1-x)^2 = 1 - 2x + x^2$$

$$B^2_1 = 2x(1-x) = 2x - 2x^2$$

$$B^2_2 = x^2$$

$$\rightarrow \quad M^{Id}_{TS} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \cancel{2}\,1 & 0 & \cancel{2}\,1 & \cancel{1}\,\tfrac{1}{2} & 0 \\ 0 & \cancel{-2}\,0 & 1 & \cancel{-1}\,1 & 0\,1 & 1 \end{array}\right)$$

$$\Rightarrow M^{Id}_{ST} = \left(M^{Id}_{TS}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

- $$M^L_{TT} M^{Id}_{TS} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix} = M^L_{TS}$$

- $$M^L_{SS} = M^{Id}_{ST} M^L_{TS} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

<u>Check:</u> $(B^2_0)' = -2 + 2x$ not direct $\downarrow$ $= -2B^2_0 - 1B^2_1$

represents

$$M^L_{SS}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

represents $-2B^2_0 - 1B^2_1 = -2(1-2x+x^2) - (2x-2x^2)$

$= -2 + 4x \cancel{-2x^2} - 2x \cancel{+2x^2}$

$= -2 + 2x$

can check other basis vectors similarly.

$$(B^2_0)' = -2 + 2x = a B^2_0 + b B^2_1 + c B^2_2$$

$$= a(1-2x+x^2) + b(2x-2x^2) + c(x^2)$$

$$= 1\cdot a + x(-2a+2b) + x^2(a-2b+c)$$

Compare coefficients: $-2 = 1\cdot a = a$

$2 = -2(-2) + 2b \Rightarrow -2 = 2b \Rightarrow b = -1$

$0 = (-2) + (-2)(-1) + c \Rightarrow c = 0$

$$\Rightarrow 1^{st} \text{ column of } M^L_{SS} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

Elementary Matrices

Elementary row transformations as used in Gaussian elimination can be viewed as left-multiplication by a matrix.

Type 1: exchange two rows of a matrix

$$P_{ij} = \begin{array}{cl} \begin{pmatrix} I & & & & \\ & 0 & & 1 & \\ & & I & & \\ & 1 & & 0 & \\ & & & & I \end{pmatrix} & \begin{array}{l} \\ i \\ \\ j \\ \\ \end{array} \\ \begin{array}{ccccc} & i & & j & \end{array} & \end{array}$$

check: $$P_{ij}\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_j \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ x_i \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix}$$

$$P_{ij}^{-1} = P_{ij}$$

Type 2: Multiply row i by non-zero scalar α

$$M_i(\alpha) = \begin{array}{cl} \begin{pmatrix} I & & \\ & \alpha & \\ & & I \end{pmatrix} & \begin{array}{l} \\ i \\ \\ \end{array} \\ \begin{array}{ccc} & i & \end{array} & \end{array}$$

$$M_i(\alpha)\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ \alpha x_i \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix}$$

$$M_i^{-1}(\alpha) = M_i\left(\tfrac{1}{\alpha}\right)$$ recall $\alpha \neq 0$

Type 3: Add the α-multiple of row j to row i

$$A_{ij}(\alpha) = \begin{array}{cl} \begin{pmatrix} I & & & \\ & \ddots & & \\ & \alpha & 1 & \\ & & & I \end{pmatrix} & \begin{array}{l} \\ \\ i \\ \\ \end{array} \\ \begin{array}{cccc} & j & i & \end{array} & \end{array}$$

$$A_{ij}(\alpha)\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_i + \alpha x_j \\ x_{i+1} \\ \vdots \\ x_n \end{pmatrix}$$

$$A_{ij}(\alpha)^{-1} = A_{ij}(-\alpha)$$ (try for yourself)

Def.: A is in <u>row-echelon form</u> (REF) if

(i) the first non-zero entry in each row is strictly to the right of the first non-zero entry of the row above. These elements are called <u>pivots</u>.

[(ii) all non-zero rows are above zero rows]

E.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is REF}$$

$$\begin{pmatrix} 5 & 1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix} \text{ is REF}$$

$$\begin{pmatrix} 2 & 3 & 1 & 5 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is not REF}$$

a matrix is in <u>reduced REF</u> (RREF), if, in addition

(iii) All pivots equal 1

(iv) The pivot is the only non-zero element in its column.

E.g.

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is RREF}$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is NOT RREF}$$

Fact: Any matrix can be brought into RREF by elementary row operations, i.e. by left-multiplying elementary matrices. (Gauss algorithm)

Def.: If $B = E_1 \cdots E_k A$, then "B is row-equivalent to A"

Theorem: $A \in M_{n\times n}(\mathbb{F})$ is invertible $\Leftrightarrow$ A is "row equivalent" to I.

Proof: "$\Rightarrow$": Let A be invertible.

We know there is $B = \underbrace{E_1 \cdots E_k}_{\text{elementary matrices}} A$ such that B is in RREF

$$\Rightarrow \quad BA^{-1} = E_1 \cdots E_k$$

$$\Rightarrow \quad BA^{-1}E_k^{-1} \cdots E_1^{-1} = I$$

Claim: $B = I$. Because: suppose not.

Since B is RREF, it must have a zero row.

But then $B \cdot *$ has a zero row, so it cannot be I $\lightning$

So A is row-equivalent to I

"$\Leftarrow$": $I = E_1 \cdots E_k A$

$$\Rightarrow \quad \underbrace{E_k^{-1} \cdots E_1^{-1}}_{\text{invertible}} = A \qquad \Rightarrow A \text{ is invertible} \qquad \square$$

and $A^{-1} = E_1 \cdots E_k$, which justifies the "usual" procedure to invert matrices.