$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ $\qquad$ $\|x\|^2 = \langle x, x \rangle$

$\langle x, y \rangle = 0 \iff x \perp y$ $\qquad$ Remark: $\mathbb{F} = \mathbb{R}$, $\quad \cos \varphi = \dfrac{\langle x, y \rangle}{\|x\| \, \|y\|}$

Examples: ① $L^2([0,2\pi]; \mathbb{R})$, $\quad f(x) = \sin x$, $\quad g(x) = \cos x$

$$\langle f, g \rangle = \int_0^{2\pi} f(x) \, g(x) \, dx = \underline{\int_0^{2\pi} \sin x \cos x \, dx}$$

$$= \underbrace{- \cos^2 x \Big|_0^{2\pi}}_{0} - \underbrace{\int_0^{2\pi} (-\cos x)(-\sin x) \, dx}_{\underline{\int_0^{2\pi} \sin x \cos x \, dx}}$$

$$\Rightarrow \int_0^{2\pi} \sin x \cos x \, dx = 0$$

$\Rightarrow \cos x \perp \sin x$

② $L^2([-1,1]; \mathbb{R})$, $\quad f(x) = x$, $\quad g(x) = x^2$

$$\langle f, g \rangle = \int_{-1}^{1} x \cdot x^2 \, dx = \int_{-1}^{1} x^3 \, dx = 0 \;\leftarrow \text{by symmetry}$$

$\Rightarrow x \perp x^2$

$h(x) = 1$, $\quad \langle h, g \rangle = \int_{-1}^{1} 1 \cdot x^2 \, dx > 0$ $\qquad$ $1 \not\perp x^2$

$1 \perp x$

③ $L^2([0,2\pi]; \mathbb{R})$ $\qquad f(x) = \sin x$

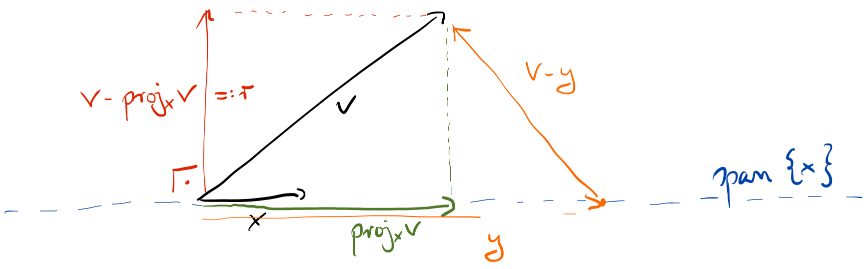
$$\|f\|^2 = \int_0^{2\pi} \underbrace{\sin^2 x}_{= \frac{1}{2}(1 - \cos 2x)} \, dx = \int_0^{2\pi} \left(\frac{1}{2} - \frac{\cos 2x}{2} \right) dx = \frac{1}{2} 2\pi - 0 = \pi$$

$\|f\| = \sqrt{\pi}$

Def. Projector : Fix $x \in V$, for any $v \in V$, define

$$\mathrm{proj}_x v = \frac{x \langle x, v \rangle}{\|x\|^2}$$

Remark: $\mathrm{proj}_x v$ depends only on span$\{x\}$, not on $\|x\|$, or a scalar multiple.



Properties: (i) $\mathrm{proj}_x \circ \mathrm{proj}_x = \mathrm{proj}_x$

(ii) $r = v - \mathrm{proj}_x v \perp x$

(iii) $\mathrm{proj}_x v$ minimizes dist$(v, \text{span}\{x\})$, i.e. among all vectors $y \in \text{span}\{x\}$, x is the vector such that $\|v-y\|$ is minimal

Proof:
$$\|v-y\|^2 = \| \underbrace{v - \mathrm{proj}_x v}_{=r} + \underbrace{\mathrm{proj}_x v - y}_{\in \text{span}\{x\}} \|^2$$
two perpendicular vectors

$$= \underbrace{\|v - \mathrm{proj}_x v\|^2}_{\text{fixed}} + \underbrace{\|\mathrm{proj}_x v - y\|^2}_{\text{smallest, in fact zero, if } y \text{ is chosen as } \mathrm{proj}_x v}$$ □

Def: $\mathcal{C} = \{x_i\}_{i \in J}$ is an orthonormal set, if

$$\langle x_i, x_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

E.g.: $\{e_1, \dots, e_n\} \subset \mathbb{R}^n$ is an orthonormal set in $\mathbb{R}^n$.

$\langle x, y \rangle = x^T y$

Theorem: $\mathcal{C} = \{x_i\}_{i \in \mathcal{I}}$ on. set, $x = \sum_{j=1}^{m} a_j x_j$ $\quad y = \sum_{j=1}^{m} b_j x_j$

(i) $a_i = \langle x_i, x \rangle$

$$\langle x_i, x \rangle = \langle x_i, \sum_{j=1}^{m} a_j x_j \rangle = \sum_{j=1}^{m} a_j \underbrace{\langle x_i, x_j \rangle}_{\delta_{ij}} = a_i$$

(ii) $\langle x, y \rangle = a^H b$

$\uparrow$ inner prod. in $\mathbb{V}$ $\qquad \uparrow$ standard inner product in $\mathbb{F}^n$

$a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$$= a_1 \underbrace{\langle x_i, x_1 \rangle}_{\substack{0 \text{ if } i \neq 1 \\ 1 \text{ if } i = 1}} + a_2 \underbrace{\langle x_i, x_2 \rangle}_{\substack{0 \text{ if } i \neq 2 \\ 1 \text{ if } i = 2}} + \dots$$

Idea of projector generalises to subspaces!

$\mathcal{C} = \{x_1, \dots, x_m\}$ on. set $\qquad \mathbb{X} = \operatorname{span} \mathcal{C} \subset \mathbb{V}$ (subspace)

$$\operatorname{proj}_{\mathbb{X}} v = \sum_{j=1}^{m} \operatorname{proj}_{x_j} v = \sum_{i=1}^{m} x_i \langle x_i, v \rangle$$

$\langle x, y \rangle = x^H y$ on $\mathbb{F}^n$

$= (\quad)\begin{pmatrix} \\ \\ \end{pmatrix}$ "inner product"

In the special case $\mathbb{V} = \mathbb{F}^n$:

$$\operatorname{proj}_{\mathbb{X}} v = \sum_{i=1}^{m} x_i (x_i^H v) = \underbrace{\left(\sum_{i=1}^{m} x_i x_i^H \right)} v$$

"outer product" $\swarrow$

$$= \begin{pmatrix} \\ \\ \end{pmatrix} (\qquad) + \begin{pmatrix} \\ \\ \end{pmatrix} (\; \dots \;)$$

$$= \underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{n} \Big\} n$$

We see that the projector can be written as matrix multiplication with the $n \times n$ - matrix $\sum_{i=1}^{m} x_i x_i^H$

Properties: (As above!)

(i) $\operatorname{proj}_{\mathbb{X}} \circ \operatorname{proj}_{\mathbb{X}} = \operatorname{proj}_{\mathbb{X}}$

(ii) $r = v - \operatorname{proj}_{\mathbb{X}} v \perp \mathbb{X}$ (i.e. $r \perp y$ for any $y \in \mathbb{X}$)

(iii) $\operatorname{proj}_{\mathbb{X}} v$ is the vector y from $\mathbb{X}$ that minimizes $\|v - y\|$

Def: $L: V \to W$ orthonormal, if

$$\langle x, y\rangle_V = \langle Lx, Ly\rangle_W \qquad \forall x, y \in V$$

Prop: $L: V \to V$ orthonormal, $\dim V < \infty$ $\Rightarrow$ L isomorphism

Proof: • L is injective because: Let $x \in N(L)$

$$\Rightarrow \quad \begin{array}{c}\langle x, x\rangle_V \\ \| \\ \|x\|_V^2\end{array} = \langle Lx, Lx\rangle_W = \|\underset{=0}{Lx}\|_W^2 = 0$$

$\Rightarrow$ $x = 0$

$\Rightarrow$ $N(L) = \{0\}$ (works even if $\dim V = \infty$)

• Rank-nullity theorem:

$$\underbrace{\dim V}_{n} = \dim R(L) + \underbrace{\dim N(L)}_{=0}$$

$\Rightarrow$ $\dim R(L) = n$

$\Rightarrow$ $R(L) = V$

$\Rightarrow$ L is surjective

Remark: If $\dim V = \infty$, then orthonormal maps $V \to V$ are injective, but not always surj.

E.g.: $V = \ell^2$, L the right-shift operator:

$$L(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

Def: $Q \in M_{n \times n}(\mathbb{F})$ is orthonormal, if it is the matrix representation of an an orthonormal operator $L: \mathbb{F}^n \to \mathbb{F}^n$ w.r.t. standard basis and standard inner prod.

Theorem: Q orthonormal $\Leftrightarrow$ $Q^H Q = I$ (so $Q^{-1} = Q^H$)

Remark: "Usual" language: Q orthonormal on $\mathbb{R}$: "Q orthogonal"

Q " on $\mathbb{C}$: "Q unitary"