Mathematics for Data Science

Mock Exam

January 26, 2026

1. Consider the subspace $V \subset C([0,1];\mathbb{R})$ spanned by the basis $B = [1, \mathrm{e}^x]$ and endowed with inner product

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, \mathrm{d}x\,.$$

   (a) Consider the linear operator $Lp = p'$. State nullspace and range of L; no computation required.

   (b) State the rank-nullity theorem and show that it applies in this example.

   (c) Find the matrix representing L with respect to basis B.

   (d) Find an orthonormal basis for V.

   (e) Find the matrix representing the change of coordinates from basis B to your orthonormal basis.

   (f) Find the matrix representing L with respect to your orthonormal basis.

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2. Let $V = M_2(\mathbb{R})$. Fix two vectors $u, v \in \mathbb{R}^2$ and define

$$f(A) = u^T A v\,.$$

   (a) Show that $f\colon V \to \mathbb{R}$ is a linear transformation.

   (b) Recall the definition of the Frobenius inner product on V,

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij}\, b_{ij} = \operatorname{tr}(A^T B)\,.$$

   and set

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

   Find a matrix $B \in V$ such that

$$f(A) = \langle A, B \rangle\,.$$

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3. Let $V = M_n(\mathbb{F})$ with $n \geq 2$. The determinant is a map $\det\colon V \to \mathbb{F}$.

   (a) Show that det is not a linear transformation.

   (b) Give an argument that det is differentiable.

   Hint: Computing the derivative is possible, but it is not easy. Better use a short (!) general argument.

   (c) Show that $D\det(0) = 0$.

   Note: This formula is using the symbol 0 in two different meanings. As part of your answer, you should state explicitly what each of the zero symbols means, i.e., which space it belongs to.

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4. Are the following statements true or false? If true, state a brief explanation or name the relevant theorem. A full proof is not required. If false or incomplete, modify the statement so that it becomes true, and give a short comment on your modification.

   (a) A normed vector space is a metric space.

   (b) A set can be both open and closed.

   (c) A set is compact if it is closed and bounded.

   (d) A sequence, together with its limit, is a closed set.

   (e) The rational numbers are a closed set.

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5. (a) Find the second-order Taylor polynomial of the function

$$f(x, y) = (2 - x - y)^3$$

   at the point $(1, 1)$.

   (b) On $C([0, \infty), \mathbb{R})$, consider the (nonlinear) map

$$T(f)(t) = \int_0^t \ln f(s)\, \mathrm{d}s\,.$$

   Compute the derivative $DT(f)[g](t)$.

   Note: A formal computation suffices – you do not need to prove that your result indeed satisfies the properties of the Fréchet derivative. You may assume that $f(t) \geq 1$ for all $t \in [0, \infty)$.

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6. Let X be a Banach space, $A\colon X \to X$ a linear operator with operator norm $\|A\| < 1$, and $b \in X$.

   Show that $f(x) = Ax + b$ has a fixed point. (10)

7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1-function such that $f(5,-2,1) = 0$ and

$$Df(5,-2,1) = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix} .$$

(a) Show that there exists an open interval $I = (1-\delta, 1+\delta)$ and a differentiable function $g\colon I \to \mathbb{R}^2$ such that $g(1) = (5,-2)$ and $f(g_1(z), g_2(z), z) = 0$ for all $z \in I$.

(b) Compute $Dg(1)$.

(c) Does there exist a differentiable function h defined on some neighborhood of $x = 5$ with values in $\mathbb{R}^2$ such that $f(x, h_1(x), h_2(x)) = 0$?

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