1. Consider the subspace $V \subset C([0,1];\mathbb{R})$ spanned by the basis $B = [1, \mathrm{e}^x]$ and endowed with inner product

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, \mathrm{d}x\,.$$

(a) Consider the linear operator $Lp \doteq p'$. State nullspace and range of L; no computation required.

(b) State the rank-nullity theorem and show that it applies in this example.

(c) Find the matrix representing L with respect to basis B.

(d) Find an orthonormal basis for V.

(e) Find the matrix representing the change of coordinates from basis B to your orthonormal basis.

(f) Find the matrix representing L with respect to your orthonormal basis.

(2+2+2+3+3+3)

(a) $\mathcal{N}(L) = \operatorname{span}\{1\}$

$R(L) = \operatorname{span}\{e^x\}$

(b) $\dim \mathcal{N}(L) + \dim R(L) = \dim V$ (If $\dim V < \infty$)

<u>Here</u>: $1 + 1 = 2$

(c) $M^L_{B,B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $q_1 = 1$ (Note that $\|q_1\|^2 = \int_0^1 1^2\, dx = 1$)

$\tilde{q}_2 = e^x - q_1 \langle q_1, e^x \rangle = e^x - \int_0^1 e^x\, dx = e^x - (e-1)$

$= e^x + 1 - e$

(solution ctd.)

$$\|\tilde{q}_2\|^2 = \int_0^1 \left(e^x + 1 - e\right)^2 dx$$

$$= \int_0^1 e^{2x} + 2(1-e)e^x + (1-e)^2 \, dx$$

$$= \tfrac{1}{2} e^{2x} \Big|_0^1 + 2(1-e)\, e^x \Big|_0^1 + (1-e)^2$$

$$= \tfrac{1}{2}\underbrace{(e^2 - 1)}_{=(e-1)(e+1)} + 2(1-e)(e-1) + (1-e)^2$$

$$= (e-1)\left[\tfrac{1}{2}e + \tfrac{1}{2} + 2 - 2e + e - 1\right]$$

$$= (e-1)\left(\tfrac{3}{2} - \tfrac{1}{2}e\right)$$

$$\Rightarrow \quad q_2 = \frac{1}{\sqrt{(e-1)\frac{1}{2}(3-e)}}\, \tilde{q}_2 \qquad \Rightarrow \quad B' = [q_1, q_2]$$

(e) $$M^{I}_{B',B} = \begin{pmatrix} 1 & c(1-e) \\ 0 & c \end{pmatrix} \qquad \text{with} \quad c = \frac{1}{\sqrt{(e-1)\frac{1}{2}(3-e)}}$$

For the inverse transformation,

$$\left(\begin{array}{cc|cc} 1 & c(1-e) & 1 & 0 \\ 0 & c & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & c(1-e) & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{c} \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 1 & -c(1-e)/c \\ 0 & 1 & 0 & \frac{1}{c} \end{array}\right)$$

$$\Rightarrow \quad M^{I}_{\mathcal{B},\mathcal{B}'} = \begin{pmatrix} 1 & (e-1) \\ 0 & \frac{1}{c} \end{pmatrix}$$

$$(f) \quad M^{L}_{\mathcal{B}',\mathcal{B}'} = M^{I}_{\mathcal{B},\mathcal{B}'} \; M^{L}_{\mathcal{B},\mathcal{B}} \; M^{I}_{\mathcal{B}',\mathcal{B}}$$

$$= \begin{pmatrix} 1 & e-1 \\ 0 & \frac{1}{c} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c(1-e) \\ 0 & c \end{pmatrix}}_{= \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & c(e-1) \\ 0 & 1 \end{pmatrix}$$

<u>Check</u> (not required):

$$q_2' = c\left(e^x + 1 - e\right)' = c e^x$$

$$= c\left(e^x + 1 - e\right) - c(1-e)$$

$$= q_2 + c(e-1)\, q_1$$

✓

2. Let $V = M_2(\mathbb{R})$. Fix two vectors $u, v \in \mathbb{R}^2$ and define

$$f(A) = u^T A v\,.$$

(a) Show that $f\colon V \to \mathbb{R}$ is a linear transformation.

(b) Recall the definition of the Frobenius inner product on V,

$$\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij}\, b_{ij} = \mathrm{tr}(A^T B)\,. \quad = tr(AB^T)$$

and set

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Find a matrix $B \in V$ such that

$$f(A) = \langle A, B\rangle\,.$$

(5+5)

(a) $f(A+B) = u^T(A+B)v = u^T(Av + Bv)$

$= u^T A v + u^T B v = f(A) + f(B) \qquad \forall A, B \in V$

$f(\lambda A) = u^T(\lambda A)v = \lambda u^T A v = \lambda f(A) \qquad \forall \lambda \in \mathbb{R}, A \in V$

(b) Solution 1 (direct computation):

$f(A) = \sum_{i,j} u_i a_{ij} v_j = a_{11}\cdot 2 + a_{12}(-1) + a_{21}\cdot 2 + a_{22}(-1)$

$= \sum_{i,j} a_{ij} b_{ij}$ with $B = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$

Solution 2: (using properties of the trace):

$f(A) = tr(u^T A v) = tr(A v u^T) = tr(AB^T) = \langle A, B\rangle$

because $u^T A v$ is a scalar

with $B = u v^T = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$

3. Let $V = M_n(\mathbb{F})$ with $n \geq 2$. The determinant is a map $\det\colon V \to \mathbb{F}$.

   (a) Show that det is not a linear transformation.

   (b) Given an argument that det is differentiable.
   Hint: Computing the derivative is possible, but it is not easy. Better use a short (!) general argument.

   (c) Show that $D\det(0) = 0$.
   Note: This formula is using the symbol 0 in two different meanings. As part of your answer, you should state explicitly what each of the zero symbols means, i.e., which space it belongs to.

(5+5+5)

(a) Counter example:

$$\det(2I) = 2^n \det I = 2^n$$

$$2 \det I = 2 \neq 2^n \quad \text{for } n \geq 2$$

(b) The determinant is a polynomial of order n of the elements of the matrix. Polynomials are differentiable (any number of times).

(c) $D\det(0) = 0$ ← 0 as a linear transformation $M_n(\mathbb{F}) \to \mathbb{F}$

$0 \in M_n(\mathbb{F})$

Let's compute the directional derivative $D_H \det(0)$:

$$D_H \det(0) = \lim_{\lambda \to 0} \frac{\det(0 + \lambda H) - \det(0)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{\lambda^n \det H}{\lambda} = \lim_{\lambda \to 0} \lambda^{n-1} \underbrace{\det H}_{\text{const}} = 0$$

Since every directional derivative is 0, $D\det(0) = 0$.

4. Are the following statements true or false? If true, state a brief explanation or name the relevant theorem. A full proof is not required. If false or incomplete, modify the statement so that it becomes true, and give a short comment on your modification.

   (a) A normed vector space is a metric space.

   (b) A set can be both open and closed.

   (c) A set is compact if it is closed and bounded.

   (d) A sequence, together with its limit, is a closed set.

   (e) The rational numbers are a closed set.

(2+2+2+2+2+2)

(a) True, with $d(x,y) = \|x-y\|$

(b) True, the empty set and the entire metric space are always open and closed.

(c) In general false, but true for finite-dimensional normed vector spaces (Heine-Borel-theorem)

(d) True: the limit of the sequence is the only limit point, and it is included

(e) False: the real numbers are the completion of the rationals.

5. (a) Find the second-order Taylor polynomial of the function

$$f(x, y) = (2 - x - y)^3$$

at the point $(1, 1)$.

(b) On $C([0,\infty), \mathbb{R})$, consider the (nonlinear) map

$$T(f)(t) = \int_0^t \ln f(s)\, ds\,.$$

Compute the derivative $DT(f)[g](t)$.

Note: A formal computation suffices – you do not need to prove that your result indeed satisfies the properties of the Fréchet derivative. You may assume that $f(t) \geq 1$ for all $t \in [0, \infty)$.

(5+5)

(a) Solution 1:

$$f(x,y) = \left((1-x) + (1-y)\right)^3$$
$$= (1-x)^3 + 3(1-x)^2(1-y) + 3(1-x)(1-y)^2 + (1-y)^3$$

So all terms are cubic in $1-x$, $1-y$

$\Rightarrow$ the second-order Taylor polynomial is zero.

Solution 2: $f(1,1) = (2-1-1)^3 = 0$

$$Df = \left(-3(2-x-y)^2, -3(2-x-y)^2\right) \Rightarrow Df(0,0) = (0,0)$$

$$\text{Hess} f = \begin{pmatrix} +6(2-x-y) & 6(2-x-y) \\ 6(2-x-y) & 6(2-x-y) \end{pmatrix} \Rightarrow \text{Hess} f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So the second-order Taylor polynomial is zero.

(b) $$\delta T(f)(t) = \int_0^t \frac{\delta f(s)}{f(s)}\, ds \Rightarrow DT(f)[g](t) = \int_0^t \frac{g(s)}{f(s)}\, ds$$

6. Let X be a Banach space, $A\colon X \to X$ a linear operator with operator norm $\|A\| < 1$, and $b \in X$.

   Show that $f(x) = Ax + b$ has a fixed point. (10)

$$\|f(x) - f(y)\| = \|Ax + b - Ay - b\|$$

$$= \|A(x-y)\| \leq \|A\| \, \|x-y\|$$

Since $\|A\| < 1$, f is a contraction on the closed set X, so the contraction mapping theorem applies and shows that f has a unique fixed point.

<u>Note:</u> Here, f maps X into X because A is linear and X is a vector space, so there is nothing to prove here.

For nonlinear f defined on a closed set D, it is important to verify that $f(D) \subset D$ when applying the CMT.

7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1-function such that $f(5,-2,1) = 0$ and

$$Df(5,-2,1) = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$

(a) Show that there exists an open interval $I = (1-\delta, 1+\delta)$ and a differentiable function $g\colon I \to \mathbb{R}^2$ such that $g(1) = (5,-2)$ and $f(g_1(z), g_2(z), z) = 0$ for all $z \in I$.

(b) Compute $Dg(1)$.

(c) Does there exist a function h defined on some neighborhood of $x = 5$ with values in $\mathbb{R}^2$ such that $f(x, h_1(x), h_2(x)) = 0$?

(5+5+5)

(a) We write $f(x,z)$ with $x \in \mathbb{R}^2$

Then $D_1 f(5,-2,1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is clearly invertible,

so the implicit function theorem applies and asserts the existence of $g \in C^1(I;\mathbb{R}^2)$ s.t. $f(g(z),z) = 0$

(b) $D(f(g(z),z)) = 0 \Rightarrow D_1 f(g(z),z)\, Dg(z) + D_2 f(g(z),z) = 0$

$\Rightarrow Dg(z) = -D_1 f(g(z),z)^{-1} D_2 f(g(z),z)$.

Here: $D_1 f(5,-2,1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow D_1 f(5,-2,1)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix}$

$\Rightarrow Dg(1) = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(c) We would need that, writing $f = f(x,y)$ with $x \in \mathbb{R}$, $y \in \mathbb{R}^2$

$D_2 f(5,-2,1) = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}$ is invertible, but it is not.

$\Rightarrow$ implicit function theorem does not apply!