Mathematics for Data Science

Final Exam

February 19, 2026

1. Consider the subspace $V \subset C([0, 2\pi]; \mathbb{C})$ spanned by $B = [1, \mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}]$ and endowed with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)}\, g(x)\, \mathrm{d}x\,.$$

When answering the following, Euler's formula

$$\mathrm{e}^{\mathrm{i}z} = \cos z + \mathrm{i}\, \sin z$$

may be helpful.

   (a) Verify that $\langle \cdot, \cdot \rangle$ is an inner product.

   (b) Show that the basis B is orthonormal. To limit the number of required computations, full credit if you check the inner products $\langle 1, 1\rangle$, $\langle 1, \mathrm{e}^{\mathrm{i}x}\rangle$, $\langle \mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{\mathrm{i}x}\rangle$, and $\langle \mathrm{e}^{-\mathrm{i}x}, \mathrm{e}^{\mathrm{i}x}\rangle$.

   (c) Consider the linear operator $Lp = p''$. State nullspace and range of L; no computation required. Then state the rank-nullity theorem and show that it applies in this example.

   (d) Find the matrix representing L with respect to basis B.

   (e) Show that $B' = [1, \cos x, \sin x]$ is also a basis for V by writing down the matrix representing the change of basis from B to B'.

   (f) Find the matrix representing L with respect to basis B'.

   (g) Let $\ell\colon V \to \mathbb{C}$ be defined by $\ell(f) = f'(0)$. Show that this is a linear transformation.

   (h) Find a function $h \in V$ such that for every $f \in V$,

$$\ell(f) = \langle h, f \rangle\,.$$

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2. Are the following statements true or false? Give a *brief* explanation in each case. No credit for a true/false guess without explanation!

   (a) Every metric space has a basis.

   (b) The closure of a set can be open.

(c) The unit ball in $\mathbb{R}^n$ is compact.

(d) The unit ball in $L^2([0,1];\mathbb{R})$ is compact.

(e) In a Banach space, every Cauchy sequence converges.

(2+2+2+2+2)

3. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is not differentiable at the origin. (5)

4. (a) Find the second-order Taylor polynomial of the function

$$f(x,y) = \mathrm{e}^{x+y^2}$$

at the point $(0,0)$.

(b) On $V = M_n(\mathbb{F})$, consider the (nonlinear) map

$$f(A) = A^{-1}$$

Show that when A is invertible,

$$Df(A)B = -A^{-1}BA^{-1}\,.$$

(5+5)

5. Let X be a Banach space and $B \in \mathcal{B}(X)$ a bounded linear operator on X with operator norm $\|B\| < \frac{1}{2}$.

Consider the map $f\colon \mathcal{B}(X) \to \mathcal{B}(X)$ defined by

$$f(A) = ABA$$

(a) Show that f is not a linear transformation but $f(A)$ is a linear transformation. (On which space?)

(b) Show that f maps the closed unit ball in $\mathcal{B}(X)$ into itself.

(c) Show that f has a fixed point on the closed unit ball in $\mathcal{B}(X)$.

(5+5+5)

6. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x,y,z) = \begin{pmatrix} x+y+z \\ xyz - y - z^2 \end{pmatrix}\,.$$

(a) Show that there exists an open interval $I = (-\delta, \delta)$ and a differentiable function $g\colon I \to \mathbb{R}^2$ such that $g(0) = (0,0)$ and $f(g_1(z), g_2(z), z) = 0$ for all $z \in I$.

(b) Compute $Dg(0)$.

(c) Does there exist a differentiable function h defined on some neighborhood of $y = 0$ with values in $\mathbb{R}^2$ such that $f(h_1(y), y, h_2(y)) = 0$?

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