1. Consider the subspace $V \subset C([0,2\pi];\mathbb{C})$ spanned by $B = [1, e^{ix}, e^{-ix}]$ and endowed with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\, g(x)\, dx\,.$$

When answering the following, Euler's formula

$$e^{iz} = \cos z + i \sin z$$

may be helpful.

(a) Verify that $\langle \cdot, \cdot \rangle$ is an inner product.

(b) Show that the basis B is orthonormal. To limit the number of required computations, full credit if you check the inner products $\langle 1,1\rangle$, $\langle 1, e^{ix}\rangle$, $\langle e^{ix}, e^{ix}\rangle$, and $\langle e^{-ix}, e^{ix}\rangle$. (5+5)

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(a) (i) $\langle f,f\rangle = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx \geq 0$

Clearly, the integral is strictly positive if f cont. and $f \neq 0$.

(ii) $\langle f, g+h\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\,(g(x)+h(x))\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} \overline{f}(x)\, g(x)\,dx + \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\, h(x)\,dx = \langle f,g\rangle + \langle f,h$

$\langle f, ag\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\; a\, g(x)\,dx$

$= a\frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\cdot g(x)\,dx = a\langle f,g\rangle$

(iii) $\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}\, g(x)\,dx = \overline{\frac{1}{2\pi}\int_0^{2\pi} f(x)\,\overline{g(x)}\,dx} = \overline{\langle g,f\rangle}$

(b) $\langle 1,1\rangle = \frac{1}{2\pi}\int_0^{2\pi} dx = 1$, $\langle 1, e^{ix}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{ix}\,dx = \frac{1}{2\pi i} e^{ix}\Big|_0^{2\pi} = 0$

$\langle e^{ix}, e^{ix}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-ix} e^{ix}\,dx = \frac{1}{2\pi}\int_0^{2\pi} dx = 1$, $\langle e^{-ix}, e^{+ix}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{2ix}\,dx = 0$

(c) Consider the linear operator $Lp = p''$. State nullspace and range of L; no computation required. Then state the rank-nullity theorem and show that it applies in this example.

(d) Find the matrix representing L with respect to basis B.

(e) Show that $B' = [1, \cos x, \sin x]$ is also a basis for V by writing down the matrix representing the change of basis from B to B'.

(f) Find the matrix representing L with respect to basis B'. (5+5+5+5)

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(c) $N(L) = \text{span}\{1\}$, $R(L) = \text{span}\{e^{ix}, e^{-ix}\}$

$\dim V = \dim N(L) + \dim R(L)$

$3 = 1 + 2$

(d)

$$M^{L}_{B,B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

since $(1)'' = 0$, $(e^{ix})'' = i^2 e^{ix} = -e^{ix}$,

$(e^{-ix})'' = (-i)^2 e^{ix} = -e^{-ix}$

(e) $e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$$\Rightarrow \quad M^{I}_{B',B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix}$$

Thus, B' is a basis as every vector from B can be represented as a l.c. of vectors from B' and the cardinality of B' agrees with B.

(f) By direct inspection,

$$M^{L}_{B',B'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

since $(1)'' = 0$,

$(\cos x)'' = -\cos x$

$(\sin x)'' = -\sin x$

(g) Let $\ell\colon V \to \mathbb{C}$ be defined by $\ell(f) = f'(0)$. Show that this is a linear transformation.

(h) Find a function $h \in V$ such that for every $f \in V$,

$$\ell(f) = \langle h, f \rangle .$$

(5+5)

(g) $\ell(f+g) = (f+g)'(0) = (f'+g')(0) = f'(0) + g'(0) = \ell(f) + \ell(g)$

$\ell(af) = (af)'(0) = (af')(0) = a\, f'(0) = a\, \ell(f)$

(h) Write $f(x) = \sum_{i=1}^{3} f_i\, b_i(x)$ with $b_1(x) = 1$, $b_2(x) = e^{ix}$, $b_3(x) = e^{-ix}$

Since $[b_1, b_2, b_3]$ are an ONB, $f_i = \langle b_i, f \rangle$

$$\Rightarrow \ell(f) = f'(0) = \sum_{i=1}^{3} f_i\, b_i'(0)$$

$$= \sum_{i=1}^{3} \langle b_i, f \rangle\, b_i'(0)$$

$$= \langle \underbrace{\sum_{i=1}^{3} \overline{b_i'(0)}\, b_i}_{=:\,h}, f \rangle$$

Here: $b_1'(0) = 0$, $b_2'(0) = i e^{ix}\big|_{x=0} = i$, $b_3'(0) = -i e^{ix}\big|_{x=0} = -i$

$\Rightarrow h(x) = -i e^{ix} + i e^{-ix} = -i(\cos x + i \sin x) + i(\cos x - i \sin x) = 2 \sin x$

2. Are the following statements true or false? Give a *brief* explanation in each case. No credit for a true/false guess without explanation!

   (a) Every metric space has a basis.
   (b) The closure of a set can be open.
   (c) The unit ball in $\mathbb{R}^n$ is compact.
   (d) The unit ball in $L^2([0,1];\mathbb{R})$ is compact.
   (e) In a Banach space, every Cauchy sequence converges.

(2+2+2+2+2)

(a) This is nonsense:
A metric space does not need to be a vector space, so it is generally not possible to even define a basis.

(b) True. E.g. for a metric space X, $\overline{X} = X$ and X is open.

(c) True. By Heine-Borel, closed and bounded sets in a finite-dimensional normed vector space (like $\mathbb{R}^n$) are compact.

(d) False. Heine-Borel does not apply because $L^2([0,1];\mathbb{R})$ is infinite-dimensional (e.g. it contains polynomials of arbitrary degree as subspace).

(e) True. Banach spaces are closed by definition, so every Cauchy sequence converges.

3. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is not differentiable at the origin. (5)

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2}\frac{\cancel{x^2}}{\cancel{x^2}} \to \frac{1}{2} \quad \text{as } x \to 0, \ x \neq 0$$

So f is not continuous at $(0,0)$, hence cannot be differentiable

4. (a) Find the second-order Taylor polynomial of the function

$$f(x,y) = e^{x+y^2}$$

at the point $(0,0)$.

(b) On $V = M_n(\mathbb{F})$, consider the (nonlinear) map

$$f(A) = A^{-1}$$

Show that when A is invertible,

$$Df(A)B = -A^{-1}BA^{-1}.$$

(5+5)

(a) $f(0,0) = e^0 = 1$

$Df = \left(e^{x+y^2},\ 2y e^{x+y^2}\right) \Rightarrow Df(0,0) = (1,0)$

$D^2f = \begin{pmatrix} e^{x+y^2} & 2y e^{x+y^2} \\ 2y e^{x+y^2} & 2e^{x+y^2} + 4y^2 e^{x+y^2} \end{pmatrix} \Rightarrow D^2f(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\Rightarrow f(x,y) = f(0,0) + Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)D^2f(0,0)\begin{pmatrix} x \\ y \end{pmatrix} + O(\|(x,y)\|^3)$

$= 1 + (1,0)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \dots$

$= 1 + x + \frac{1}{2}x^2 + y^2$

<u>Or:</u> $e^{x+y^2} = 1 + (x+y^2) + \frac{1}{2}(x+y^2)^2 + O(|x+y^2|^3) = 1 + x + y^2 + \frac{1}{2}x^2 + \dots$

(b) $f(A)A = I$

$\Rightarrow \delta f A + f \delta A = 0$

$\Rightarrow \delta f = -f \delta A A^{-1}$

$\Rightarrow Df(A)\delta A = -A^{-1}\delta A A^{-1}$

5. Let X be a Banach space and $B \in \mathcal{B}(X)$ a bounded linear operator on X with operator norm $\|B\| < \frac{1}{2}$.

Consider the map $f \colon \mathcal{B}(X) \to \mathcal{B}(X)$ defined by

$$f(A) = ABA$$

(a) Show that f is not a linear transformation but $f(A)$ is a linear transformation. (On which space?)

(b) Show that f maps the closed unit ball in $\mathcal{B}(X)$ into itself.

(c) Show that f has a fixed point on the closed unit ball in $\mathcal{B}(X)$.

(5+5+5)

(a) $f(2A) = (2A)B(2A) = 4ABA \neq 2f(A)$

$\Rightarrow$ f is not linear as a map on $\mathcal{B}(X)$.

But $f(A) = ABA$ is a composition of (bounded) linear operators on X, so $f(A)$ is again a (bounded) linear operator on X.

(b) $\|f(A)\| = \|ABA\| \leq \|A\| \|B\| \|A\| < \frac{1}{2}\|A\|^2 \leq \frac{1}{2}$ if $\|A\| \leq 1$

So f maps the unit ball into a ball of radius $\frac{1}{2}$.

(c) Clearly, $f(0) = 0B0 = 0$ (the zero-operator)

So 0 is a fixed point and it's contained in the closed unit ball.

If you did not see this, you can use the CMT:

$$\begin{aligned}
\|f(A) - f(\tilde{A})\| &= \|ABA - \tilde{A}B\tilde{A}\| \\
&= \|(A-\tilde{A})BA + \tilde{A}B(A-\tilde{A})\| \\
&\leq \underbrace{\|A\|}_{\leq 1} \|B\| \|A-\tilde{A}\| + \underbrace{\|\tilde{A}\|}_{\leq 1} \|B\| \|A-\tilde{A}\| \\
&\leq \kappa \|A-\tilde{A}\| \quad \text{with } \kappa = 2\|B\| < 1.
\end{aligned}$$

So f is a contraction on the closed unit ball, and maps it into itself $\Rightarrow$ CMT is applicable.

6. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x,y,z) = \begin{pmatrix} x+y+z \\ xyz - y - z^2 \end{pmatrix}.$$

(a) Show that there exists an open interval $I = (-\delta, \delta)$ and a differentiable function $g\colon I \to \mathbb{R}^2$ such that $g(0) = (0,0)$ and $f(g_1(z), g_2(z), z) = 0$ for all $z \in I$.

(b) Compute $Dg(0)$.

(c) Does there exist a differentiable function h defined on some neighborhood of $y = 0$ with values in $\mathbb{R}^2$ such that $f(h_1(y), y, h_2(y)) = 0$?

(5+5+5)

(a) $Df = \begin{pmatrix} 1 & 1 & 1 \\ yz & xz-1 & -2z \end{pmatrix} \Rightarrow Df(0,0,0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Splitting variables into $x_1 = \begin{pmatrix} x \\ y \end{pmatrix}$, $x_2 = z$

we see that $D_1 f(0,0,0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ which is clearly invertible

with inverse $D_1 f(0,0,0)^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, so the implicit function theorem applies and yields the quoted result. Moreover:

(b) $Df(g(z), z) = 0 \Rightarrow (D_1 f)(g(z), z)\, Dg + (D_2 f)(g(z), z) = 0$

$\Rightarrow Dg(0) = -D_1 f(0,0,0)^{-1} D_2 f(0,0,0)$

$= -\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

(c) In this case, $D_1 f$ is the matrix Df with the middle column removed, so the implicit function theorem does not apply.