Mathematics for Data Science

Semester Review

Winter Semester 2025/2026

1. Basic notions of Linear Algebra (vector space, linearity, matrix representation of linear maps, change of coordinates, range, nullspace, inner product, norm, orthogonality, matrix and operator norms).

2. Properties of trace and determinant. E.g., can you show the following: Let $p_A(z) = \det(zI - A)$ denote the characteristic polynomial of a matrix $A \in M_n(\mathbb{F})$. Then

$$p_A(z) = z^n - \operatorname{tr}(A)\, z^{n-1} + p_{n-2}(z)$$

   where p_{n-2} is a polynomial of degree at most $n - 2$.

3. Diagonalization: Basic idea, normal matrices as a special case (see Week 6 homework).

4. Singular value decomposition: Know what it is, and the properties of the associated matrix. Can you relate the rank-nullity theorem to the SVD? Can you relate the least-square solution of a linear system to the SVD? (See, in particular, Exercises 4.29 and 4.32.)

5. Basic topology: Understand the definitions of open set, closed set, compact set, boundary of a set, interior point, limit point. No detailed proofs, but basic definitions and concepts might be tested.

6. Limits, uniform limits, continuity, uniform continuity, also in connection with differentiability (see next item).

   There is a large number of examples in the exercises that illustrate these concepts (all of Week 9 exercises; for differentiability, see Week 10). An example of possible questions is the following statement: *Show that a small perturbation of an invertible matrix remains invertible.* Can you make this statement more explicit by fixing the notions of vector space, norm, neighborhood? Can you give a quick proof using basic topology? Can you give a proof that provides explicit estimates?

7. Definition and basic properties of the key notions of derivative: Fréchet derivative (or total derivative), directional derivative and its relation to the Fréchet derivative, partial derivative, Jacobian. Be able to compute the Fréchet derivative in the following situations:

(a) For maps $\mathbb{R}^n \to \mathbb{R}^m$ given by an explicit expression. In this case, compute the Jacobian (like Example 6.2.13),

(b) For maps defined on vector spaces of functions such as $C([0,1];\mathbb{R})$ (like Example 6.3.5),

(c) For matrix expressions (e.g., show that for $f(A) = A^{-1}$ defined on the vector space of invertible $n \times n$ matrices, $Df(A)B = -A^{-1}BA^{-1}$).

8. Taylor's theorem: be able to compute Taylor polynomials in simple cases (Examples 6.6.11, 6.6.12, and Exercises 6.18–20).

   Note: The point at which the Taylor polynomial is computed does not need to be 0 (as in almost all the examples in the book)! You do not need to remember the formula for the remainder in Taylor's theorem, but potentially work with it in simple cases when it is given.

9. Contraction mapping theorem: Know the assumptions (f maps a closed set into itself, and is a contraction) and be able to verify them in simple cases (see Exercises 7.2, 7.8).

10. Implicit and inverse function theorems.

    (a) Computational questions like Examples 7.4.3, 7.4.5, 7.4.6, and Exercises 7.18–21.

    Note: Depending on the number of variables, such problems typically involve checking a matrix for invertibility and/or computing the inverse matrix.

    (b) Abstract conceptual questions: Exercise 7.23; can you use the inverse function theorem to prove the implicit function theorem, or vice versa?