

Exercise 10 - Solutions

$$\begin{array}{r}
 1. \quad 10001011 \\
 \quad 11111000 \\
 \quad 00000000 \\
 \text{XOR } 01011010 \\
 \hline
 \quad 00101001 \quad \leftarrow \text{Reconstructed bit sequence on failed disk}
 \end{array}$$

2. First, we prove the hint.

$$\text{Given: } a \equiv a' \pmod{97}$$

This means that $a = a' + n \cdot 97$ for some $n \in \mathbb{Z}$

$$\begin{aligned}
 \Rightarrow 10^k a + b &= 10^k a' + 10^k \cdot n \cdot 97 + b \\
 &= 10^k a' + b + m \cdot 97 \quad \text{for some } m \in \mathbb{Z}
 \end{aligned}$$

$$\Rightarrow 10^k a + b \equiv 10^k a' + b \pmod{97}$$

Now we want to compute $i \pmod{97}$ for some big integer i .

$$\begin{array}{l}
 \text{Fix } k \text{ and write } i = 10^k a + b. \\
 \begin{array}{c} \uparrow \quad \uparrow \\ \text{all digits up to decimal digit } k+1 \quad \text{last } k \text{ digits of } i \end{array}
 \end{array}$$

$$(\text{E.g. } k=3, i=286793 \text{ can be written } i = 10^3 \cdot 286 + 793.)$$

Then compute $a' := a \pmod{97}$, i.e. a' has at most two decimal digits.

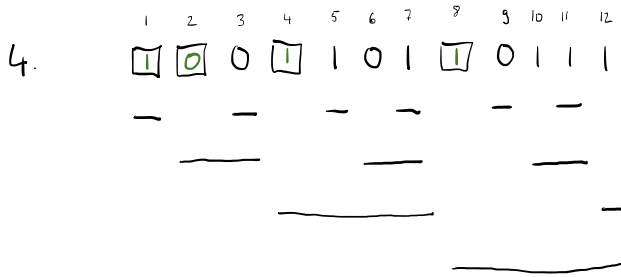
$$\begin{aligned}
 \text{Then, by the hint, } i &\equiv \underbrace{10^k a'} + b \pmod{97} \\
 &=: i'
 \end{aligned}$$

So we can continue to compute $i' \pmod{97}$, where i' is potentially shorter than i as it has at most $k+2$ decimal digits.

$$3(a). \quad \text{DE5175...} \rightarrow 750303000007633300 \underbrace{131451}_{\substack{\text{D} \\ \text{E}}} =: i \quad \text{Now } i \equiv 1 \pmod{97}, \text{ so code is } \underline{\text{valid}}$$

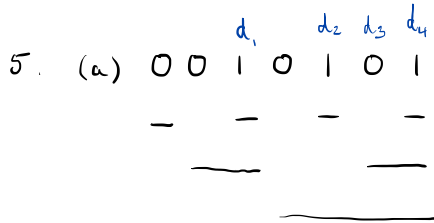
(In Python, compute $i \% 97$)

$$(b) \quad \text{Similarly, } 750303000007363300131451 = 36 \pmod{97}, \text{ so code is } \underline{\text{invalid}}.$$



parity bits green

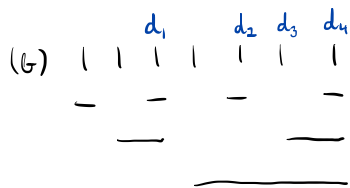
data bits black



$p_1 = 1$
 $p_2 = 0$
 $p_4 = 0$

} There is an error at position $(001)_2 = 1$

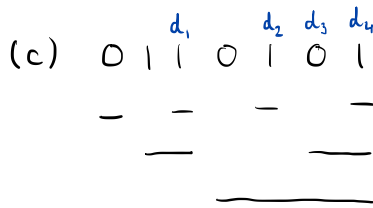
The correct message is 1101



$p_1 = 0$
 $p_2 = 0$
 $p_4 = 0$

} The code is correct.

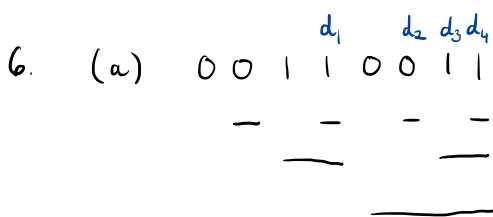
The correct message is 1111



$p_1 = 1$
 $p_2 = 1$
 $p_4 = 0$

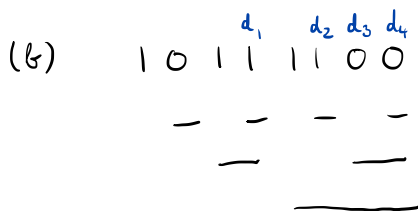
} The code has an error at $(011)_2 = 3$

The correct message is 0101



$p_0 = 0$, so overall parity is O.K.
 $p_1 = 0$
 $p_2 = 0$
 $p_4 = 0$

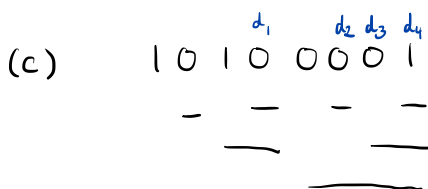
} Code correct, message is 1011



$p_0 = 1$, assume single-bit error
 $p_1 = 0$
 $p_2 = 0$
 $p_4 = 0$

} all O.K., so single-bit error must be in p_0

Message is 1100



$p_0 = 1$, assume single-bit error
 $p_1 = 1$
 $p_2 = 0$
 $p_4 = 1$

} Error at $(101)_2 = 5$ (at data bit d_2)

Message is 0101

(d) 00110101 $p_0 = 0$, so overall parity O.K.

| | | | | | |
|---|---|---|---|---|--|
| - | - | - | - | } | indicates error, which must be 2-bit Discard message. |
| — | — | — | — | | |
| — | — | — | — | | |
| — | — | — | — | | |

$p_1 = 1$
 $p_2 = 1$
 $p_4 = 0$

7. The errors follow a binomial distribution, with $p = \frac{1}{100}$, $q = 1-p = \frac{99}{100}$

(a) $P(0 \text{ errors in 4 bits}) = \binom{4}{0} p^0 q^{4-0} = \left(\frac{99}{100}\right)^4 \approx 0.961$

(b) $P(0 \text{ or } 1 \text{ errors in 7 bits}) = \binom{7}{0} p^0 q^{7-0} + \binom{7}{1} p^1 q^{7-1} \approx 0.9980$

(c) Assume that re-transmission is 100% successful, i.e. a detected two-bit error will result in a correct message. Since the true value is ~~very~~ close to 100%, the error in this approximation is practically negligible. So we need to compute:

$$P(0, 1 \text{ or } 2 \text{ errors in 8 bits}) = \binom{8}{0} p^0 q^{8-0} + \binom{8}{1} p^1 q^{7-1} + \binom{8}{2} p^2 q^{7-2} \approx 0.99997$$