Exercise 6 – Solutions

1. (a) $\{\xi_n = j,\ \xi_1 \neq j, \dots, \xi_{n-1} \neq j\} \subset \{\xi_n = j\}$

$$\Rightarrow \underbrace{P(\xi_n = j,\ \xi_1 \neq j, \dots, \xi_{n-1} \neq j \mid \xi_0 = i)}_{= f_n(j|i)} \leq \underbrace{P(\xi_n = j \mid \xi_0 = i)}_{= p_n(j|i)}$$

so the statement is true.

(b) $$\{\xi_n = j\} = \bigcup_{k=1}^{n} \{\xi_n = j,\ \xi_k = j,\ \xi_1 \neq j, \dots, \xi_{k-1} \neq j\}$$

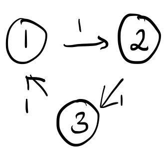
is a partition into mutually exclusive and collectively exhaustive events.

$$\Rightarrow \underbrace{P(\xi_n = j \mid \xi_0 = i)}_{= p_n(j|i)} = \sum_{k=1}^{n} \underbrace{P(\xi_n = j,\ \xi_k = j,\ \xi_1 \neq j, \dots, \xi_{k-1} \neq j \mid \xi_0 = i)}_{= \underbrace{P(\xi_n = j \mid \xi_k = j)}_{= p_{n-k}(j|j)} \underbrace{P(\xi_k = j,\ \xi_1 \neq j, \dots, \xi_{k-1} \neq j \mid \xi_0 = i)}_{= f_k(j|i)}}$$

by Markov property

So the statement is true.

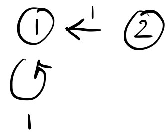
(c) False. E.g.:



- all states intercommunicate
- $p(2|1) = 1$
- $p(1|2) = 0$ (But: $p_2(1|2) = 1$)

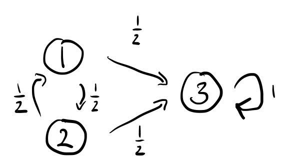
(d) False. E.g.

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$ represents



so 1 is recurrent and 2 is transient.

(e) False. E.g.



Here, $1 \Leftrightarrow 2$, but $\underbrace{\sum_{k=1}^{\infty} f_k(1|2)}_{\text{probability that 2 ever transitions to 1}} = \sum_{k=1}^{\infty} f_k(2|1) = \frac{1}{2} < 1$

(f) Suppose that i is transient. By criterion from class, this implies

$$\sum_{n=0}^{\infty} p_n(i|i) < \infty$$

$$\Rightarrow \quad p_n(i|i) \xrightarrow[n \to \infty]{} 0 \qquad \text{which contradicts } p_n(i|j) \longrightarrow \delta_{ij}$$

So statement is true.

2. (a) We can model the process with the tuple (r_n, b_n) describing the number of red balls and the number of blue balls in urn 1. Then

$$S = \{ (r_n, b_n) : r_n, b_n \in \{0, \dots, 10\} \}$$

so that

$$|S| = 11^2 = 121$$

(b) Let $X_{(r,b)}$ denote the probability that urn 1 has r red balls and b blue balls.

Then, initially,

$$X_{(10,0)} = 1 \quad \text{and} \quad X_{(r,b)} = 0 \quad \text{for } (r,b) \neq (10,0)$$

(c) Suppose urn 1 is selected.

$$p((\hat{r},\hat{b}) \mid (r,b)) = \begin{cases} \delta_{\hat{r}0}\,\delta_{\hat{b}0} & \text{if } (r,b) = (0,0) \\ \delta_{\hat{r},r-1}\,\delta_{\hat{b},b}\,\frac{r}{r+b} + \delta_{\hat{r}r}\,\delta_{\hat{b},b-1}\,\frac{b}{r+b} & \text{otherwise} \end{cases}$$

Suppose urn 2 is selected.

$$p((\hat{r},\hat{b}), (r,b)) = \begin{cases} \delta_{\hat{r},10}\,\delta_{\hat{b},10} & \text{if } (r,b) = (10,10) \\ \delta_{\hat{r},r+1}\,\delta_{\hat{b}b}\,\frac{10-r}{20-r-b} + \delta_{\hat{r},r}\,\delta_{\hat{b},b+1}\,\frac{10-b}{20-r-b} & \text{otherwise} \end{cases}$$

To obtain the full transition matrix:

- Take the symmetric average of the above
- To implement the stopping criterion, make the final state $(0,10)$ absorbing, i.e., set

$$p((\hat{r},\hat{b}),(0,10)) = \delta_{\hat{r},0}\,\delta_{\hat{b},10}.$$

(d) State $(0,10)$ is absorbing, all other states connect to $(0,10)$, so they are transient.

Note: If we do not stop the game, all states are interconnected.

A finite-state-space Markov chain must have at least one recurrent state, so all states are then recurrent.

3. (a)

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) $T = \{1\}$

$R_1 = \{2,3\}$ irreducible, closed, recurrent

$R_2 = \{4,5,6\}$ " " "

(c) $d(1) = d(4) = d(5) = d(6) = 1$, $d(2) = d(3) = 2$

(d) $f_n(3|1) = \left(\frac{1}{2}\right)^{n-2}\frac{1}{3}$ for $n \geq 2$, $f_1(3|1) = 0$

(e) Need to solve $Px = x \Rightarrow (P-I)x = 0$ Augmented matrix:

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the solution is spanned by the normalized eigenvectors

$$\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{2}{5}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$$

These basis vectors represent the invariant density of each recurrent class, every convex combination is an invariant density for the full system.