Exercise 5 - Solutions

1(a) $\quad \zeta_n = \left(\frac{q}{p}\right)^{\xi_n} \qquad \xi_n = \eta_1 + \dots + \eta_n \qquad P(\eta_i = 1) = p \; ; \; P(\eta_i = -1) = q = 1-p$

$$\mathbb{E}[\zeta_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{\xi_n} \left(\frac{q}{p}\right)^{\eta_{n+1}} \mid \mathcal{F}_n\right]$$

$$= \left(\frac{q}{p}\right)^{\xi_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{\eta_{n+1}}\right] \quad \text{(taking out what is known; independence)}$$

$$= \zeta_n \left(\underbrace{\left(\frac{q}{p}\right)^1 p + \left(\frac{q}{p}\right)^{-1} q}_{= q+p = 1} \right)$$

$$= \zeta_n$$

(b) $\quad \Theta_n = C^n \lambda^{\xi_n}$

$$\mathbb{E}[\Theta_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[C^{n+1} \lambda^{\xi_n} \lambda^{\eta_{n+1}} \mid \mathcal{F}_n\right]$$

$$= C^{n+1} \lambda^{\xi_n} \mathbb{E}[\lambda^{\eta_{n+1}}] \quad \text{(taking out what is known; independence)}$$

$$= \Theta_n \, C \, (\lambda p + \lambda^{-1} q)$$

For Θ_n to be a Martingale, need $C(\lambda p + \lambda^{-1} q) = 1$

$$\Rightarrow C = \frac{1}{\lambda p + \frac{1-p}{\lambda}} = \frac{\lambda}{\lambda^2 p + 1 - p} = \frac{\lambda}{(\lambda^2 - 1)p + 1}$$

2(a) Let $\eta_i \in \{-1, 1\}$ be the amount won or lost each round

$\xi_n = \eta_1 + \dots + \eta_n$ the cumulated winnings/losses, $\xi_0 := 0$

$\tau = \min\{n : \xi_n = a \text{ or } \xi_n = -b\}$ the stopping time.

The stopped process ξ_τ is a martingale (because stopping is a gambling strategy, and a gambling strategy applied to a martingale is also a martingale).

We apply the optional stopping theorem to ξ_τ (for complete justification, see class!)

$$\Rightarrow \quad 0 = \mathbb{E}[\xi_0] = \mathbb{E}[\xi_\tau]$$

$$= -b \underbrace{P(\xi_\tau = -b)}_{=:p} + a \underbrace{P(\xi_\tau = a)}_{=1-p}$$

$$\Rightarrow \quad a - ap - bp = 0 \quad \Rightarrow \quad p = \frac{a}{a+b} = P(\xi_\tau = -b)$$

$$1-p = 1 - \frac{a}{a+b} = \frac{b}{a+b} = P(\xi_\tau = a)$$

(b) $\quad \zeta_n = \xi_n^2 - n$ is a martingale (shown in class), so ζ_τ is a martingale.

Apply optional stopping (for full justification, see class)

$$0 = \mathbb{E}[\zeta_0] = \mathbb{E}[\zeta_\tau^2] = \mathbb{E}[\xi_\tau^2 - \tau]$$

$$\Rightarrow \mathbb{E}[\tau] = \mathbb{E}[\xi_\tau^2] = b^2 p + a^2 (1-p) = \frac{b^2 a}{a+b} + \frac{a^2 b}{a+b} = ab$$

3. Use $\zeta_n = \left(\frac{q}{p}\right)^{\xi_n}$, which is a martingale according to 1(a).

Process stops if $\zeta_n = \left(\frac{q}{p}\right)^{-b}$ or $\zeta_n = \left(\frac{q}{p}\right)^{a}$. By optional stopping,

$$1 = \mathbb{E}[\zeta_0] = \mathbb{E}[\zeta_\tau]$$

$$= \left(\frac{q}{p}\right)^{-b} \underbrace{P(\xi_\tau = -b)}_{=:r} + \left(\frac{q}{p}\right)^{a} \underbrace{P(\xi_\tau = a)}_{=1-r}$$

$$\Rightarrow \quad 1 = \left(\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^{a}\right) r + \left(\frac{q}{p}\right)^{a}$$

$$\Rightarrow \quad r = \frac{1 - \left(\frac{q}{p}\right)^{a}}{\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^{a}}$$

$$1 - r = \frac{\left(\frac{q}{p}\right)^{-b} - 1}{\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^{a}}$$

To compute the expected stopping time, we cannot proceed as above because $S_n^2 - n$ is NOT a martingale (look at the proof from class of this result).

However, we can construct another martingale: Set

$$\Theta_n = \xi_n + cn$$

$$\Rightarrow \mathbb{E}[\Theta_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\xi_n + \eta_{n+1} + c(n+1) \mid \mathcal{F}_n]$$

$$= \xi_n + cn + c + \underbrace{\mathbb{E}[\eta_{n+1}]}_{= -1\cdot q + 1\cdot p} \quad \text{(taking out what is known, indep.)}$$

$$= \Theta_n \qquad \text{if we set } c := q-p$$

Then, by optional stopping,

$$0 = \mathbb{E}[\Theta_0] = \mathbb{E}[\Theta_\tau] = \mathbb{E}[\xi_\tau + c\tau]$$

$$\Rightarrow \mathbb{E}[\tau] = -\frac{1}{c}\,\mathbb{E}[\xi_\tau] = \frac{1}{p-q}\left(-b\,P(\xi_\tau = -b) + a\,P(\xi_\tau = a)\right)$$

$$= \frac{1}{p-q}\left(-b\,\frac{1-\left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^a} + a\,\frac{\left(\frac{q}{p}\right)^{-b} - 1}{\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^a}\right)$$

$$= \frac{1}{p-q}\;\frac{b\left(\left(\frac{q}{p}\right)^a - 1\right) + a\left(\left(\frac{q}{p}\right)^{-b} - 1\right)}{\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^a}$$

4. Given: $\mathbb{E}[\eta_{n+1} \mid \mathcal{F}_n] = a_n \eta_n + b_n$

$$\xi_n := c_n \eta_n + d_n$$

$$\Rightarrow \mathbb{E}[\xi_{n+1} \mid \mathcal{F}_n] = c_{n+1}\,\underbrace{\mathbb{E}[\eta_{n+1} \mid \mathcal{F}_n]}_{= a_n \underbrace{\eta_n}_{= \frac{1}{c_n}(\xi_n - d_n)} + b_n} + d_{n+1}$$

$$= c_{n+1}\frac{a_n}{c_n}\xi_n - c_{n+1}\frac{a_n}{c_n}d_n + c_{n+1}b_n + d_{n+1}$$

So for ξ_{n+1} to be a martingale, need

$$c_{n+1}\frac{a_n}{c_n} = 1 \Rightarrow c_{n+1} = \frac{c_n}{a_n}$$

and

$$-\underbrace{c_{n+1}\frac{a_n}{c_n}}_{=1} d_n + \underbrace{c_{n+1}}_{=\frac{c_n}{a_n}} b_n + d_{n+1} = 0$$

$$\Rightarrow d_{n+1} = d_n - \frac{c_n}{a_n} b_n$$

We can set $c_0 = 1 \Rightarrow c_n = \prod_{i=0}^{n-1}\frac{1}{a_i}$

$$d_0 = 0 \Rightarrow d_n = -\sum_{i=0}^{n-1} c_{i+1} b_i$$

5.

$$\eta_{n+1} = \begin{cases} \eta_n - 1 & \text{if ball was removed from urn 1 and placed elsewhere, } p = \frac{\eta_n}{N}\frac{K-1}{K} \\ \eta_n + 1 & \text{" " " " " other urn " " in urn 1, } p = \frac{N-\eta_n}{N}\frac{1}{K} \\ \eta_n & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathbb{E}\left[\eta_{n+1} \mid \eta_n\right] = (\eta_n - 1)\frac{\eta_n}{N}\frac{K-1}{K} + (\eta_n+1)\frac{N-\eta_n}{N}\frac{1}{K} + \eta_n\left(1 - \frac{\eta_n}{N}\frac{K-1}{K} - \frac{1-\eta_n}{N}\frac{1}{K}\right)$$

cancellation

$$= -\frac{\eta_n}{N}\frac{K-1}{K} + \frac{N-\eta_n}{N}\frac{1}{K} + \eta_n$$

$$= -\frac{\eta_n}{N} + \frac{1}{K} + \eta_n$$

$$= \frac{N-1}{N}\eta_n + \frac{1}{K}$$

Use Exercise 4 with $a_n = \frac{N-1}{N}$, $b_n = \frac{1}{K}$

$$\Rightarrow c_n = \left(\frac{N}{N-1}\right)^n$$

$$d_n = -\sum_{i=0}^{n-1}\left(\frac{N}{N-1}\right)^{i+1}\frac{1}{K} = -\frac{N}{N-1}\frac{1}{K}\sum_{i=0}^{n-1}\left(\frac{N}{N-1}\right)^i$$

$$= -\frac{N}{N-1}\frac{1}{K}\frac{\left(\frac{N}{N-1}\right)^n - 1}{\frac{N}{N-1} - 1} = -\frac{1}{K}\frac{\left(\frac{N}{N-1}\right)^n - 1}{1 - \frac{N-1}{N}} = \frac{N}{K}\left(1 - \left(\frac{N}{N-1}\right)^n\right)$$