Exercise 4 – Solutions

1 (a). $\mathbb{E}[\eta \mid \xi_1] = \mathbb{E}[4\xi_1\xi_2 \mid \xi_1]$

$= 4\xi_1\, \mathbb{E}[\xi_2 \mid \xi_1]$ as ξ_1 is $\sigma(\xi_1)$-measurable

$= 4\xi_1\, \mathbb{E}[\xi_2]$ as ξ_1 and ξ_2 are independent

$= 4\xi_1\,(1\cdot p + 0\cdot(1-p))$

$= 4p\xi_1$

(b) If $\omega \in \zeta^{-1}(\{1\}) = \{\xi_1+\xi_2 = 0\} = \{00\}$:

$$\mathbb{E}[\xi_1 \mid \zeta](\omega) = 0$$

If $\omega \in \zeta^{-1}(\{0\}) = \{\xi_1+\xi_2 \neq 0\} = \{01, 10, 11\}$:

$$\mathbb{E}[\xi_1 \mid \zeta](\omega) = 0\cdot\frac{p(1-p)}{1-(1-p)^2} + 1\cdot\frac{p(1-p)}{1-(1-p)^2} + 1\,\frac{p^2}{1-(1-p)^2}$$

$$= \frac{p}{2p-p^2} = \frac{1}{2-p}$$

We can write the answer also more compactly using characteristic functions:

$$\mathbb{E}[\xi_1 \mid \zeta] = \underbrace{\mathbb{E}[\xi_1 \mid \{\zeta=0\}]}_{=1/(2-p) \text{ as above}} \mathbb{1}_{\{\zeta=0\}} + \underbrace{\mathbb{E}[\xi_1 \mid \{\zeta=1\}]}_{=0} \mathbb{1}_{\{\zeta=1\}}$$

$$= \frac{1}{2-p}\, \mathbb{1}_{\{\zeta=0\}}$$

By symmetry, we get the same expressions for $\mathbb{E}[\xi_2 \mid \zeta]$.

(c) The two random variables $\mathbb{E}[\xi_1 | \zeta]$ and $\mathbb{E}[\xi_2 | \zeta]$ are identical a.s.

But a random variable is independent of itself if and only if it is a constant a.s., i.e.

$$\underbrace{P(\{\zeta=0\})=0}_{\Rightarrow p=0} \quad \text{or} \quad \underbrace{P(\{\zeta=0\})=1}_{p=1}$$

So, in general they are dependent, except if $p=0$ or $p=1$.

2. $$\mathbb{E}[\eta_{n+1} | \xi_1, \dots, \xi_n] = \mathbb{E}[\eta_n + \xi_{n+1} | \xi_1, \dots, \xi_n]$$

$$= \eta_n + \mathbb{E}[\xi_{n+1} | \xi_1, \dots, \xi_n] \quad \text{as } \eta_n \text{ is } \sigma(\xi_1, \dots, \xi_n)\text{-measurable.}$$

$$= \eta_n + \underbrace{\mathbb{E}[\xi_{n+1}]}_{=p \text{ by assumption}} \quad \text{by independence}$$

3. $\mathbb{E}[\eta^2] = \mathbb{E}[\xi^2]$

$$\Rightarrow \mathbb{E}[(\xi-\eta)^2] = \mathbb{E}[\xi^2] - 2\mathbb{E}[\xi\eta] + \mathbb{E}[\eta^2]$$

$$= 2\left[\mathbb{E}[\eta^2] - \mathbb{E}[\xi\eta]\right]$$

$$= 2\mathbb{E}\left[\mathbb{E}[\xi|\mathcal{G}]\left(\mathbb{E}[\xi|\mathcal{G}] - \xi\right)\right]$$

$$= 2\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}[\xi|\mathcal{G}]\left(\mathbb{E}[\xi|\mathcal{G}] - \xi\right) | \mathcal{G}\right]\right]$$

$$= 2\,\mathbb{E}\left[\mathbb{E}[\xi|\mathcal{G}]\,\underbrace{\mathbb{E}\left[\mathbb{E}[\xi|\mathcal{G}] - \xi | \mathcal{G}\right]}_{=0}\right] = 0 \quad \Rightarrow \xi = \eta \text{ a.s.}$$