Exercise 3 – Solutions

1. $M_X = e^{\lambda(e^t - 1)}$ (from class)

$\Rightarrow K_X(t) = \ln M_X(t) = \lambda(e^t - 1)$ (cumulant gen. fctn.)

mean and variance are the first and second cumulant, i.e.

$$\mathbb{E}[X] = K_X'(0) = \lambda e^t \Big|_{t=0} = \lambda$$

$$\mathrm{Var}[X] = K''(0) = \lambda e^t \Big|_{t=0} = \lambda$$

2. (a) The rate is 2 calls/h, for interval of 2h, $\lambda = 4$

PMF: $P(X=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$

$$\Rightarrow P(X=1) = \frac{4 \cdot e^{-4}}{1!} \approx 0.073$$

(b) For an interval of $\frac{1}{2}$h, the rate is $\lambda = 1$

$$P(X \geqslant 2) = 1 - P(X=1) - P(X=0)$$

$$= 1 - \frac{1^1 \cdot e^{-1}}{1!} - \frac{1^0 e^{-1}}{0!} = 1 - \frac{1}{e} - \frac{1}{e}$$

$$\approx 0.26$$

(c) Using the Poisson distribution assumes that calls are independent. In the absence of additional information, this is the best assumption we can make, and the Poisson distribution is then the appropriate model. But independence may break down for many reasons: E.g.,

- a call may generate a follow-up call
- the is an issue that hits many uses at the same time
- there are times when more people are at work, thus making support calls more likely.

3. (a)

$$\int_{\mathbb{R}^3} f_X(x)\,dx = \int_0^1\int_0^1\int_0^1 (x_1^2 + x_2^2 + x_3^2)\,dx_1\,dx_2\,dx_3$$

$$= 3\int_0^1\int_0^1\underbrace{\int_0^1 x_1^2\,dx_1}_{=\frac{1}{3}x_1^3\big|_0^1 = \frac{1}{3}}\,dx_2\,dx_3 \quad (\text{by symmetry})$$

$$= \frac{3}{3}\int_0^1 dx_2 \int_0^1 dx_3 = 1$$

(b)

$$P\left(X_1 \le \tfrac{1}{5}, X_2 \le \tfrac{1}{3}, X_3 \le \tfrac{1}{2}\right) = \int_0^{\frac{1}{5}}\int_0^{\frac{1}{3}}\int_0^{\frac{1}{2}} (x_1^2 + x_2^2 + x_3^2)\,dx_3\,dx_2\,dx_1$$

$$= \int_0^{\frac{1}{5}} x_1^2\,dx_1 \int_0^{\frac{1}{3}} dx_2 \int_0^{\frac{1}{2}} dx_3 + \int_0^{\frac{1}{5}} dx_1 \int_0^{\frac{1}{3}} x_2^2\,dx_2 \int_0^{\frac{1}{2}} dx_3 + \int_0^{\frac{1}{5}} dx_1 \int_0^{\frac{1}{3}} dx_2 \int_0^{\frac{1}{2}} x_3^2\,dx_3$$

$$= \frac{1}{3}\left(\frac{1}{5}\right)^3 \frac{1}{3}\,\frac{1}{2} + \frac{1}{5}\,\frac{1}{3}\left(\frac{1}{3}\right)^3 \frac{1}{2} + \frac{1}{5}\,\frac{1}{3}\,\frac{1}{3}\left(\frac{1}{2}\right)^3$$

$$= \frac{1}{3\cdot 3\cdot 5\cdot 2}\underbrace{\left(\frac{1}{5^2}+\frac{1}{3^2}+\frac{1}{2^2}\right)}_{} \approx 0.0045$$

$$= \frac{4\cdot 9 + 25\cdot 4 + 25\cdot 9}{25\cdot 9\cdot 4} = \frac{361}{900}$$

(c) $$P\left(X_1 \geq \tfrac{1}{5}, X_2 \leq \tfrac{1}{3}, X_3 \leq \tfrac{1}{2}\right) = \int_{\frac{1}{5}}^{1}\int_0^{\frac{1}{3}}\int_0^{\frac{1}{2}} \left(x_1^2 + x_2^2 + x_3^2\right) dx_3\,dx_2\,dx_1$$

(as before)

$$= \dots = \frac{1}{3}\left(1 - \frac{1}{5^3}\right)\frac{1}{3}\,\frac{1}{2} + \frac{4}{5}\,\frac{1}{3}\,\frac{1}{3^3}\,\frac{1}{2} + \frac{4}{5}\,\frac{1}{3^2}\,\frac{1}{2^3}$$

$$\approx 0.071$$

(d) $$\mathbb{E}[X] = \int_0^1\int_0^1\int_0^1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \left(x_1^2 + x_2^2 + x_3^2\right) dx_1\,dx_2\,dx_3$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \underbrace{\int_0^1\int_0^1\int_0^1 x_1\left(x_1^2 + x_2^2 + x_3^2\right) dx_1\,dx_2\,dx_3}_{} \quad \text{by symmetry}$$

$$= \int_0^1 x_1^3\,dx_1 \int_0^1 dx_2 \int_0^1 dx_3 + 2\int_0^1 x_1\,dx_1 \int_0^1 x_2^2\,dx_2 \int_0^1 dx_2 \quad \text{also by symmetry}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix}\left[\frac{1}{4} + 2\cdot\frac{1}{2}\frac{1}{3}\right] = \frac{7}{12}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

(e) $$f_1(x) = \int_0^1\int_0^1 (x_1^2 + x_2^2 + x_3^2)\, dx_2\, dx_3$$

$$= x_1^2 + 2\int_0^1 x_2^2\, dx_2 = \begin{cases} x_1^2 + \frac{2}{3} & \text{if } x_1 \in [0,1] \\ 0 & \text{if } x_1 \notin [0,1] \end{cases}$$

(f) When $i=j$:

$$\mathrm{Cov}(X_i, X_j) = \mathrm{Var}(X_i) = \mathrm{Var}(X_1) \quad \text{by symmetry}$$

$$= \underbrace{\int_0^1\int_0^1\int_0^1 x_1^2 (x_1^2 + x_2^2 + x_3^2)\, dx_1\, dx_2\, dx_3}_{= \int_0^1 x_1^4\, dx_1 + 2\left(\int_0^1 x_1^2\, dx_1\right)^2 = \frac{1}{5} + 2\left(\frac{1}{3}\right)^2 = \frac{9+10}{45} = \frac{19}{45}} - \left(\frac{7}{12}\right)^2$$

$$= \frac{19}{45} - \frac{49}{144} \approx 0.082$$

When $i \neq j$:

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]^2 \quad \text{by symmetry}$$

$$\Rightarrow \operatorname{Cov}(X_i, X_j) = \int_0^1\int_0^1\int_0^1 x_1 x_2 (x_1^2 + x_2^2 + x_3^2)\, dx_1\, dx_2\, dx_3 - \frac{49}{144}$$

$$= 2\int_0^1 x_1^3\, dx_1 \int_0^1 x_2\, dx_2 + \left(\int_0^1 x_1\, dx_1\right)^2 \int_0^1 x_3^2\, dx_3 - \frac{49}{144}$$

(by symmetry)

$$= 2\,\frac{1}{4}\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{3} - \frac{49}{144}$$

$$\approx -0.0070$$

4. (a) $E[X_i] = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{1}{2} =: \mu$

$\operatorname{Var}[X_i] = E\left[\left(X_i - \frac{1}{2}\right)^2\right] = E\left[\frac{1}{4}\right] = \frac{1}{4} =: \sigma^2 \qquad \Rightarrow \sigma = \frac{1}{2}$

The CLT as stated in the book (and lectures) says that

$$Z_n = \frac{X_1 + \dots + X_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow[\text{in distr.}]{} N(0,1)$$

$$= 2\,\frac{X_1 + \dots + X_n}{\sqrt{n}} - \sqrt{n}$$

(b) The PDF of $N(0,1)$ is $f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\Rightarrow E[|Z|] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|\, e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \underbrace{\int_0^\infty x e^{-\frac{x^2}{2}} \, dx}_{} \qquad \text{by symmetry}$$

$$= \int_0^\infty e^{-u} \, du = -e^{-u} \Big|_0^\infty = 1 \qquad u = \frac{x^2}{2} \Rightarrow du = x\,dx$$

$$= \sqrt{\frac{2}{\pi}}$$

(c) This suggests the following procedure:

(i) Toss a fair coin $N = n \cdot m$ times, both n and m large.

Store the result in an array

$$X_{ij} = \begin{cases} 0 & \text{coin came up T in trial } (i,j) \\ 1 & \text{coin came up H in trial } (i,j) \end{cases}$$

$$i = 1, \dots, m \qquad j = 1, \dots, n$$

(ii) For every $i = 1, \dots, m$, compute

$$Z_i = 2 \frac{X_{i1} + \dots + X_{in}}{\sqrt{n}} - \sqrt{n}$$

This yields m approximately normally distributed samples

(iii) Compute the sample expectation

$$MAD = \frac{|Z_1| + \dots + |Z_m|}{m}$$

(iv) Then $\pi \approx \frac{2}{MAD^2}$