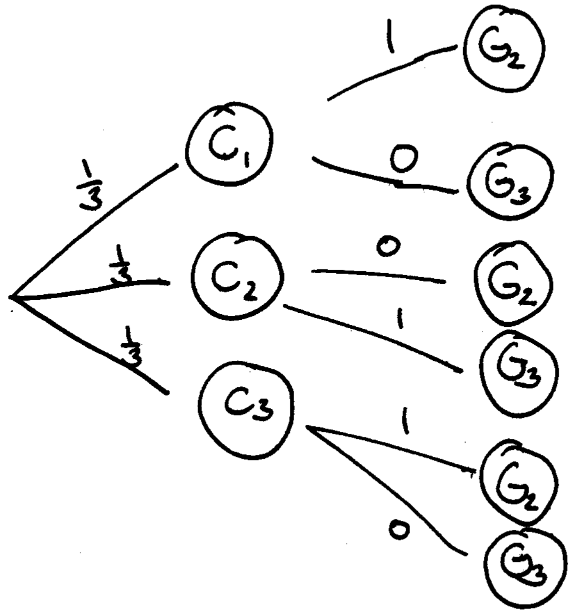
Exercise 2 - Solutions

1. Let C_i be the event that the car is behind door i,

   G_i be the event that Monty (intentionally) opens door i with a goat behind.

We can visualize this variation of the game in a probability tree, taking the convention that you initially pick door #1.



First, note that by the law of total probability

$$P(G_3) = P(G_3|C_1)P(C_1) + P(G_3|C_2)P(C_2) + P(G_3|C_3)P(C_3)$$

$$= 0 \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3} = \tfrac{1}{3}$$

$$P(G_2) = P(G_2|C_1)P(C_1) + P(G_2|C_2)P(C_2) + P(G_2|C_3)P(C_3)$$

$$= 1 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3} = \tfrac{2}{3}$$

$$\Rightarrow P(C_3 \mid G_2) = \frac{P(G_2 \mid C_3)\, P(C_3)}{P(G_2)} = \frac{1 \cdot \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

So if Monty opens door #2 and you see a goat, you may or may not switch – your probability of winning is $\frac{1}{2}$ in either case.

$$P(C_2 \mid G_3) = \frac{P(G_3 \mid C_2)\, P(C_2)}{P(G_3)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}} = 1$$

Thus, if Monty opens door #3 and you see a goat, you should switch to door #2 and will win for sure.

2 (a) P (first gets right coat) = $\frac{1}{5}$

(b) P (second gets right coat | first gets right coat) = $\frac{1}{4}$

(c) P (every guest gets the right coat) = $\frac{1}{5!}$

(there are 5! ways to permute the coats, and only one is correct)

(e) P (second gets right coat | first takes wrong coat, but not of second guest) = $\frac{1}{4}$

(it's just like (b) from the point of view of second guest)

(d) If the first guest takes the second guest's coat, their chance to get the right coat is clearly 0.

(f) P (second right) = P(second right | first takes second coat) P(first takes 2^{nd} coat)
+ P(second right | first does not take 2^{nd} coat) P(first does not take 2^{nd} coat)

$\Rightarrow P(\text{second right}) = 0 \cdot \frac{1}{5} + \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$

(This result is obvious, because if we do not know what happened with the coat of the first guest, we might as well consider the second guest the only to leave, getting one out of 5 coats.)

3. $E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$

$$= \frac{21}{6} = \frac{7}{2} = 3.5$$

$$E[X^2] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6}$$

$$= \frac{91}{6}$$

$$\Rightarrow Var[X] = \frac{91}{6} - \frac{49}{4} = \frac{182}{12} - \frac{147}{12} = \frac{35}{12} \approx 2.92$$

4. (a) Binomial distribution with $n = 100$ and $p = \frac{1}{10}$.

(b) $E[X] = np = 10$

(c) $P(X=30) = \binom{100}{30} \left(\frac{1}{10}\right)^{30} \left(\frac{9}{10}\right)^{70} \approx 1.8 \cdot 10^{-8}$