Stochastic Processes

Summer Semester 2026, Exercise 6

Due Thursday, July 2, 2026

1. Let ξ_n, $n \in \mathbb{N}$, be a Markov chain with state space S. Recall that $p_n(j|i)$ is the probability that state i transitions to state j in n steps; $f_n(j|i)$ denotes the probability that state i transitions for the first time to state j in n steps.

   Determine if the following statements are true or false. If true, provide a short argument, if false, provide a counterexample.

   (a) $p_n(j|i) \geq f_n(j|i)$ for all $n \in \mathbb{N}$ and $i, j \in S$.

   (b) $p_n(j|i) = \sum_{k=1}^{n} f_k(j|i)\, p_{n-k}(j|j)$ for $n \geq 1$.

   (c) If $i \leftrightarrow j$, there exists $n \in \mathbb{N}$ such that $p_n(i|j) > 0$ and $p_n(j|i) > 0$.

   (d) If all columns of the transition matrix are equal, then all states belong to the same class. (We take the convention that the transition matrix acts on a distribution by multiplication from the left, where the distribution vectors is a column vector.)

   (e) If $\sum_{n=1}^{\infty} f_n(j|i) < 1$ and $\sum_{n=1}^{\infty} f_n(i|j) < 1$, then i and j do not intercommunicate.

   (f) If $p_n(i|j) \to \delta_{ij}$ (the Kronecker delta) as $n \to \infty$ for all $i, j \in S$, then all states are recurrent.

2. Two urns are filled with balls. The red urn has initially 10 red balls, the blue urn has initially 10 blue balls. Each round, one urn is selected, each with probability $\frac{1}{2}$. Then, a ball from that container is selected uniformly at random and moved to the other urn. If the selected urn is empty, no ball is moved.

   The game is stopped when all red balls are in the blue urn and all blue balls are in the red urn.

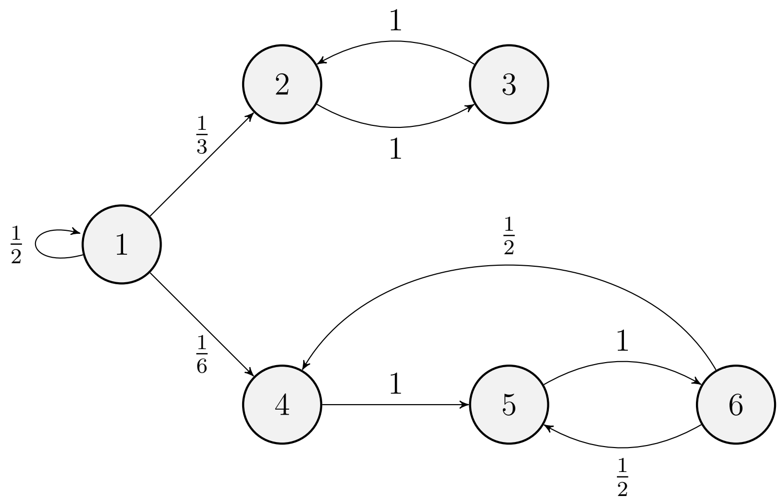
   (a) Argue that this problem can be modeled as a Markov chain. What is the state space S?

   (b) What is the initial distribution?

(c) Write out a general expression for the elements of the transition matrix, p_{ij} for $i, j \in S$.

(d) Which states are transient? Explain!

3. Consider the following Markov chain.



(a) Write out the transition matrix.

(b) Identify the classes.

(c) Find the periods of all states.

(d) Compute $f_n(3|1)$ for all $n \in \mathbb{N}$.

(e) Find the stationary distributions of the recurrent classes.