Stochastic Processes

Summer Semester 2026, Exercise 5

Due Thursday, June 18, 2026

1. Let η_i, $i = 1, 2, \dots$ be a sequence of independent random variables with $P(\eta_i = 1) = p$ and $P(\eta_i = -1) = q \equiv 1 - p$. Set

$$\xi_n = \eta_1 + \cdots + \eta_n$$

and let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ be the natural filtration.

   (a) Prove that

$$\zeta_n = \left(\frac{q}{p}\right)^{\xi_n}$$

   is a martingale with respect to the same filtration.

   (b) Given $\lambda > 0$, let

$$\theta_n = C^n \, \lambda^{\xi_n} \, .$$

   Determine the constant C such that θ_n is a martingale with respect to the same filtration.

2. A gambler wins or looses one euro in each independent round of a sequence of fair games. He stops the game when he has won a euros or lost b euros. His initial balance is zero.

   (a) What is the probability of stopping the game with a positive balance?

   (b) What is the expected stopping time?

   Note: This is very close to the example given in class, with slightly different parameters. You should base your answer on the optional stopping theorem. You do not need to verify that the conditions for the optional stopping theorem are met (this is technical and the proof from class applies almost literally).

3. Repeat the previous problem when the probability of winning in each round is p and the probability of losing in each round is $q = 1 - p$.

   Hint: Turn the process into a martingale using the trick from Problem 1, then apply optional stopping.

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$ a filtration, and $\eta_1, \eta_2, \ldots$ a sequence of integrable random variables adapted to the filtration. Assume that there exist numbers a_n and b_n such that

   $$\mathbb{E}[\eta_{n+1}|\mathcal{F}_n] = a_n \, \eta_n + b_n \, .$$

   Find two sequences c_n and d_n such that

   $$\xi_n = c_n \, \eta_n + d_n$$

   is a martingale with respect to the same filtration.

5. N balls are placed randomly in K urns. At each time step, we choose one of the balls uniformly at random and place it into one of the urns uniformly at random (so it is possible that a ball is picked up and replaced into the same urn). Let η_n denote the number of balls in the first urn at the end of time step n of this process. We consider η_n as a stochastic process with respect to its natural filtration.

   Use the construction from the previous problem to determine a process

   $$\xi_n = c_n \, \eta_n + d_n$$

   that is a martingale with respect to the same filtration.