Stochastic Processes

Summer Semester 2026, Exercise 4

Due Thursday, June 4, 2026

1. Let ξ_i, $i = 1, 2$ be independent Bernoulli-distributed random variables with parameter $p \in [0, 1]$.

   (a) Set $\eta = 4\,\xi_1\,\xi_2$. Find
   $$\mathbb{E}[\eta|\xi_1]\,.$$

   (b) Set $\zeta = \mathbf{1}_{\{\xi_1+\xi_2=0\}}$. Find
   $$\mathbb{E}[\xi_i|\zeta]\,.$$

   (c) Are $\mathbb{E}[\xi_1|\zeta]$ and $\mathbb{E}[\xi_2|\zeta]$ independent?

2. Let $\xi_1, \ldots, \xi_n$ be independent Bernoulli-distributed random variables with parameter $p \in [0, 1]$, and set
   $$\eta_n = \xi_1 + \cdots + \xi_n\,.$$
   Compute
   $$\mathbb{E}[\eta_{n+1}|\xi_1, \ldots, \xi_n]\,.$$

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let ξ be a random variable with finite variance and define
   $$\eta = \mathbb{E}[\xi|\mathcal{G}]\,.$$
   Prove that if $\mathbb{E}[\xi^2] = \mathbb{E}[\eta^2]$, then $\xi = \eta$ a.s.