Stochastic Processes

Summer Semester 2026, Exercise 3

Due Thursday, May 21, 2026

1. Compute mean and variance of a random variable X that is Poisson-distributed with rate λ.

   Hint: Recall from class that the moment generating function of the Poisson distribution is
   $$M_X(t) = \mathrm{e}^{\lambda(\mathrm{e}^t - 1)} \,.$$

2. IT support receives calls at an average rate of 2 per hour.

   (a) Assuming that calls are Poisson-distributed, compute the probability that one call is received in two hours.

   (b) Likewise, compute the probability that two or more calls are received in half an hour.

   (c) Discuss why or why not the Poisson distribution may be an appropriate model for the distribution of IT support calls.

3. (HJ Exercise 5.43.) Let $X = (X_1, X_2, X_3)$ be a multivariate random variable taking values in $\mathbb{R}^3$ with PDF defined for $\boldsymbol{x} = (x_1, x_2, x_3)$ as
   $$f_X(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|^2 & \text{if } x_i \in [0,1] \text{ for } i = 1,2,3 \\ 0 & \text{otherwise}\,. \end{cases}$$

   (a) Verify that
   $$\int_{\mathbb{R}^3} f_X(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 1 \,.$$

   (b) Find $P(X_1 \leq \frac{1}{5}, X_2 \leq \frac{1}{3}, X_3 \leq \frac{1}{2})$.

   (c) Find $P(X_1 \geq \frac{1}{5}, X_2 \leq \frac{1}{3}, X_3 \leq \frac{1}{2})$.

   (d) Find $\mathbb{E}[X]$.

   (e) Find the marginal PDF $f_1(x)$ of X_1.

   (f) Find the covariance
   $$\operatorname{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\,\mathbb{E}[X_j] \,.$$

4. Show that you can obtain an estimate of the mathematical constant π by flipping a fair coin a large number of times in the following way.

   (a) Let X_i be the Bernoulli random variable of the i-th coin flip. Argue that

   $$Z_n = 2\,\frac{X_1 + \cdots + X_n}{\sqrt{n}} - \sqrt{n}$$

   converges in distribution to the standard normal distribution.

   Hint: Use the central limit theorem.

   (b) Suppose Z is normally distributed with mean zero and variance one. Show that the *mean absolute deviation* (MAD) is given by

   $$\mathbb{E}[|Z|] = \sqrt{\frac{2}{\pi}}\,,$$

   so that

   $$\pi = \frac{2}{\mathbb{E}[|Z|]^2}\,.$$

   (c) Use this last expression to devise a procedure for estimating π. Write a short Python code that simulates this experiment.