Stochastic Processes

Mock Exam

July 6, 2026

1. Let ξ_n denote the standard symmetric random walk, i.e.,

$$\xi_n = \eta_1 + \cdots + \eta_n$$

where the η_i are independent random variables with $P(\eta_i = 1) = P(\eta_i = -1) = \frac{1}{2}$. Compute the following.

(a) $P(\xi_4 = 3 | \xi_0 = 1)$

(b) $P(\xi_4 = 3 | \xi_0 = 2)$

(5+5)

(a) To go from 1 to 3 in 4 steps, need 3 steps to the right and one step to the left, in any order. This has probability according to the Binomial distribution, of

$$p_4(3|1) = \binom{4}{1} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3} = \frac{4!}{1!\,3!} \left(\frac{1}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}.$$

(b) In an even number of steps, we always move from even to even or from odd to odd. Thus, this transition is impossible, so

$$p_4(3|2) = 0$$

2. On a probability space $(\Omega, \mathcal{F}, P)$ with sub-sigma-algebra $\mathcal{G} \subset \mathcal{F}$, we may define the conditional variance of a random variable ξ as

$$\operatorname{Var}(\xi|\mathcal{G}) = \mathbb{E}\big[(\xi - \mathbb{E}[\xi|\mathcal{G}])^2|\mathcal{G}\big] .$$

(a) Prove that

$$\operatorname{Var}(\xi) = \mathbb{E}[\operatorname{Var}(\xi|\mathcal{G})] + \operatorname{Var}(\mathbb{E}[\xi|\mathcal{G}]) .$$

(b) Let ξ_i be i.i.d. random variables with $\mathbb{E}[\xi_i] = \mu$ and $\operatorname{Var}(\xi_i) = \sigma^2$. Let ν be an independent positive-integer-valued random variable and set

$$\eta_\nu = \xi_1 + \cdots + \xi_\nu .$$

Prove that

$$\mathbb{E}[\eta_\nu] = \mu\,\mathbb{E}[\nu]$$

(Hint: Find $\mathbb{E}[\eta_\nu|\nu]$ first!), then conclude

$$\operatorname{Var}(\eta_\nu) = \mu^2 \operatorname{Var}(\nu) + \sigma^2\,\mathbb{E}[\nu]$$

(Hint: use part (a)).

(5+5)

(a) $\mathbb{E}[\operatorname{Var}(\xi|\mathcal{G})] + \operatorname{Var}(\mathbb{E}[\xi|\mathcal{G}])$

$= \mathbb{E}\big[(\xi - \mathbb{E}[\xi|\mathcal{G}])^2\big] + \mathbb{E}\big[(\mathbb{E}[\xi|\mathcal{G}] - \mathbb{E}[\xi])^2\big]$ using definitions and the tower property (twice)

$= \mathbb{E}\big[\xi^2 - 2\xi\mathbb{E}[\xi|\mathcal{G}] + \mathbb{E}[\xi|\mathcal{G}]^2 + \mathbb{E}[\xi|\mathcal{G}]^2 - 2\mathbb{E}[\xi|\mathcal{G}]\mathbb{E}[\xi] + \mathbb{E}[\xi]^2\big]$

$= \mathbb{E}[\xi^2] \underbrace{-2\mathbb{E}[\mathbb{E}[\xi|\mathcal{G}]^2]}_{\text{have used "taking out what is known" and tower property}} + 2\mathbb{E}[\mathbb{E}[\xi|\mathcal{G}]^2] \underbrace{-\mathbb{E}[\xi]^2}_{\text{have used "taking out what is known" + cancel terms}} = \operatorname{Var}(\xi)$

(b) $\mathbb{E}[\eta_\nu|\nu] = \nu\,\mathbb{E}[\xi_i] = \nu\mu$. Now take $\mathbb{E}$: $\mathbb{E}[\eta_\nu] = \mu\mathbb{E}[\nu]$

$\operatorname{Var}(\eta_\nu) \overset{(a)}{=} \mathbb{E}[\operatorname{Var}(\eta_\nu|\nu)] + \operatorname{Var}(\mathbb{E}[\eta_\nu|\nu]) = \underbrace{\mathbb{E}[\nu\sigma^2]}_{=\sigma^2\mathbb{E}[\nu]} + \underbrace{\operatorname{Var}(\nu\mu)}_{=\mu^2\operatorname{Var}(\nu)}$

3. Let ξ_n, $n \in \mathbb{N}$, be an arbitrary integrable stochastic process adapted to a filtration $\mathcal{F}_n$. For $n \geq 1$, set

$$\eta_n = \xi_n - \mathbb{E}[\xi_n | \mathcal{F}_{n-1}]$$

and further $\zeta_0 = 0$ and

$$\zeta_n = \sum_{k=1}^{n} \eta_k$$

for $n \geq 1$.

Show that ζ_n is a martingale with respect to the filtration $\mathcal{F}_n$. (5)

$$\mathbb{E}[\zeta_{n+1} | \mathcal{F}_n] = \mathbb{E}\left[\zeta_n + \xi_{n+1} - \mathbb{E}[\xi_{n+1} | \mathcal{F}_n] \mid \mathcal{F}_n\right]$$

$$= \underbrace{\zeta_n}_{\text{here we used "take out what is known"}} + \mathbb{E}[\xi_{n+1} | \mathcal{F}_n] - \underbrace{\mathbb{E}[\xi_{n+1} | \mathcal{F}_n]}_{\text{here we used the tower property}}$$

$$= \zeta_n$$

$\Rightarrow \zeta_n$ is a martingale.

4. Consider a game where a fair coin is tossed repeatedly. Gamblers can bet on predicting the next toss. If they predict correctly, they win twice their stake; if they mispredict, they lose it.

A single gambler is betting on observing the sequence HTHTHT as follows. They enter the game at some round with a stake of 1, betting on observing H. If they win, they bet their winnings on T, the next letter of the sequence; if they lose, they quit the game. So long as they are in the game, they will continue betting on each next letter, doubling their stake with each round, until they have either quit or have observed the entire sequence HTHTHT, at which point they will also quit the game, keeping their winnings.

(a) Argue that ξ_n, the process representing a gambler's winnings or losses, is a martingale.

(b) Now, at every round of the game, a new gambler enters, playing the same strategy. Argue that the process ζ_n of cumulated winnings or losses of all gamblers is a martingale.

(c) The game is stopped at a time τ when the sequence HTHTHT is observed for the first time. Argue that

$$\zeta_\tau = 2^6 + 2^4 + 2^2 - \tau\,.$$

(d) Apply the optional stopping theorem to find the expected number of rounds that need to be played before the sequence HTHTHT is observed.

(You may assume that the technical conditions for optional stopping are satisfied; extra credit if you can name and prove some or all of these conditions.)

(3+3+4+5)

(a) The process of bidding on getting the next symbol correct with a unit stake is a fair (symmetric) random walk. Changing the stake in a pre-visible way as described is a gambling strategy, so the resulting process is also a martingale.

(b) At each round, the overall process is a sum of finitely many shifts of the process from (a), so it's also a martingale by linearity of the conditional expectation.

(c) Suppose the game is stopped at time τ. Then the last 6 symbols of the sequence are HTHTHT.

The gamblers who are still in the game are:

- gambler who joined at $\tau-6$ and now won 2^6 unit stakes
- gambler who joined at $\tau-4$ and saw HTHT appear, so is currently 2^4 unit stakes on the winning side
- gambler who joined at $\tau-2$ and saw HT appear, so is currently 2^2 unit stakes on the winning side.

All other gamblers must have already quit at this point.

At time τ, altogether τ gamblers have entered the game and paid one unit stake each to play, so a total of τ unit stakes went to the casino. So total winnings/losses are

$$Z_\tau = 2^6 + 2^4 + 2^2 - \tau$$

(d) Optional stopping says $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0] = 0$

$$\Rightarrow \mathbb{E}[\tau] = 2^6 + 2^4 + 2^2$$

5. Let ξ_n be a Markov chain with states $s_1, \ldots, s_6$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 1 \end{pmatrix}.$$

(a) Find all classes of states (transient states and closed irreducible classes of recurrent states).

(b) Find all stationary distributions.

(5+5)

Solution 1: (Use Linear Algebra for everything)

To find the stationary distributions, need to solve $(P-I)x = 0$.

We proceed via elementary row transformations:

$$\begin{pmatrix} -\frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & -1 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{3}{2} & 0 \\ 0 & +1 & 0 & -1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ The nullspace of $P-I$ is spanned by the normalized vectors

$$x_1 = \frac{1}{4}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This shows that we have a closed recurrent irreducible class $C_1 = \{s_1, s_2, s_4\}$, a second such class $C_2 = \{s_6\}$ and transient states $T = \{s_3, s_5\}$.

<u>graph (optional):</u>