Stochastic Processes

Final Exam

July 20, 2026

1. Let ξ_n denote the *asymmetric* random walk
$$\xi_n = \eta_1 + \cdots + \eta_n \,,$$
where the η_i are independent random variables with $P(\eta_i = 2) = P(\eta_i = -1) = \frac{1}{2}$. Compute the following.
   (a) $P(\xi_4 = 3 | \xi_0 = 1)$
   (b) $P(\xi_4 = 3 | \xi_0 = 2)$

(5+5)

2. Let ξ_i, $i = 1, 2, \ldots$ be independent Bernoulli-distributed random variables with parameter $p \in [0, 1]$. Let ν be the number of trials until the first "success" is obtained.
   (a) Show that
   $$\mathbb{E}[\nu | \xi_1] = \xi_1 + (1 + \mathbb{E}[\nu])(1 - \xi_1) \,.$$
   (b) Show that
   $$\mathbb{E}[\nu] = \frac{1}{p} \,.$$
   (Hint: argue first that (a) implies $\mathbb{E}[\nu] = p + (1 + \mathbb{E}[\nu])(1 - p)$.)

(5+5)

3. Let ξ_i, $i \in \mathbb{N}$, be i.i.d. integrable random variables taking positive values. Set $\mu = \mathbb{E}[\xi_i]$.
   (a) Show that
   $$\eta_n = \frac{\xi_1 \, \xi_2 \ldots \xi_n}{\mu^n}$$
   is a martingale.
   (b) Find a constant α such that
   $$\zeta_n = \ln \eta_n - n \, \alpha$$
   is a martingale.

(5+5)

4. In the setting of Problem 3, suppose that $\eta_0 = 1$ and

$$\xi_i = \begin{cases} \frac{1}{2} & \text{with probability } \frac{2}{3} \\ 2 & \text{with probability } \frac{1}{3}\,. \end{cases}$$

The process is stopped when $\eta_n = \frac{1}{4}$ or $\eta_n = 4$.

(a) Argue why the optional stopping theorem is applicable. No need to prove anything, argue by analogy with a standard problem.

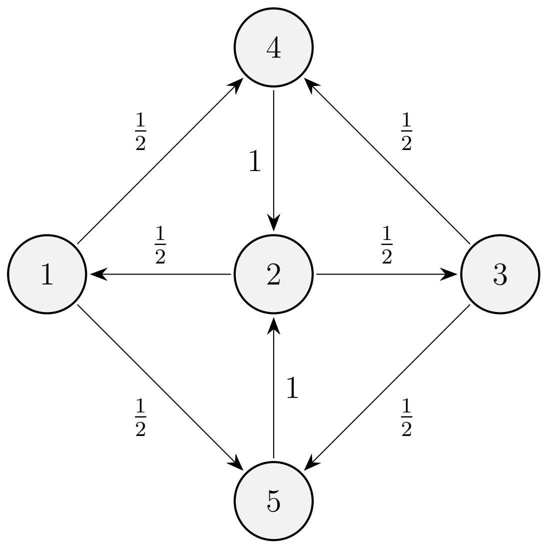
(b) Compute the probabilities that the process is stopped at $\eta_n = \frac{1}{4}$ or $\eta_n = 4$, respectively.

(c) Compute the expected stopping time.
(Hint: use Problem 3(b).)

(d) Now suppose the process is only stopped when $\eta_n = \frac{1}{2}$. Is the optional stopping theorem still applicable? Explain!

(2+3+3+2)

5. Consider the following Markov chain.



(a) Write out the transition matrix.

(b) Identify the classes.

(c) Find the periods of all states.

(d) Find the stationary distribution(s).

(2+2+2+4)