1. Let $\xi_n$ denote the *asymmetric* random walk

$$\xi_n = \eta_1 + \cdots + \eta_n\,,$$

where the $\eta_i$ are independent random variables with $P(\eta_i = 2) = P(\eta_i = -1) = \frac{1}{2}$. Compute the following.

(a) $P(\xi_4 = 3 | \xi_0 = 1)$

(b) $P(\xi_4 = 3 | \xi_0 = 2)$

(5+5)

(a) an increase by two in 4 steps can only be achieved by making two doublesteps to the right and two single steps to the left, in any order. So

$$p_4(3|1) = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} = \frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 2}\left(\frac{1}{2}\right)^4 = \frac{3}{8}$$

(b) In 4 steps, we can only achieve the increments

$-4, -1, 2, 5, 8$

$\Rightarrow \quad p_4(3|2) = 0$

2. Let $\xi_i$, $i = 1, 2, \ldots$ be independent Bernoulli-distributed random variables with parameter $p \in [0, 1]$. Let $\nu$ be the number of trials until the first "success" is obtained.

(a) Show that

$$\mathbb{E}[\nu|\xi_1] = \xi_1 + (1 + \mathbb{E}[\nu])(1 - \xi_1).$$

(b) Show that

$$\mathbb{E}[\nu] = \frac{1}{p}.$$

(Hint: argue first that (a) implies $\mathbb{E}[\nu] = p + (1 + \mathbb{E}[\nu])(1 - p)$.)

(5+5)

(a) If $\xi_1 = 1$ ("success"), then $\nu = 1$ $\Rightarrow$ $\mathbb{E}[\nu \mid \xi_1 = 1] = 1$

If $\xi_1 = 0$ ("failure"), then we need to start anew

$$\Rightarrow \mathbb{E}[\nu \mid \xi_1 = 0] = 1 + \mathbb{E}[\nu]$$

The given formula just combines both cases into a single expression.

(b) $\mathbb{E}[\nu] = \mathbb{E}[\nu \mid \xi_1 = 1]\, P(\xi_1 = 1) + \mathbb{E}[\nu \mid \xi_1 = 0]\, P(\xi_1 = 0)$

$$= 1 \cdot p + (1 + \mathbb{E}[\nu])(1-p)$$

$$= 1 + \mathbb{E}[\nu](1-p)$$

$$\Rightarrow \mathbb{E}[\nu] = \frac{1}{p}$$

3. Let $\xi_i$, $i \in \mathbb{N}$, be i.i.d. integrable random variables taking positive values. Set $\mu = \mathbb{E}[\xi_i]$.

(a) Show that

$$\eta_n = \frac{\xi_1 \xi_2 \ldots \xi_n}{\mu^n}$$

is a martingale. wrt. the natural filtration $\mathcal{F}_n$.

(b) Find a constant $\alpha$ such that

$$\zeta_n = \ln \eta_n - n\,\alpha$$

is a martingale.

(5+5)

(a) Integrability and adaptedness to natural filtration is obvious.

Let's check the martingale property:

$$\mathbb{E}[\eta_{n+1} \mid \mathcal{F}_n] = \underbrace{\mathbb{E}[\eta_n \mid \mathcal{F}_n]}_{=\eta_n} \underbrace{\mathbb{E}\left[\frac{\xi_{n+1}}{\mu} \mid \mathcal{F}_n\right]}_{=\frac{\mathbb{E}[\xi_{n+1}]}{\mu}=1} \text{ by independence}$$

("taking out what is known") by indep.

$$= \eta_n$$

(b) $$\mathbb{E}[\zeta_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[\ln \eta_n + \ln \frac{\xi_{n+1}}{\mu} - (n+1)\alpha \mid \mathcal{F}_n\right]$$

$$= \underbrace{\ln \eta_n - n\alpha}_{=\zeta_n} + \mathbb{E}\left[\ln \frac{\xi_{n+1}}{\mu}\right] - \alpha$$

(taking out what is known, independence)



So we need $\alpha = \mathbb{E}\left[\ln \frac{\xi_{n+1}}{\mu}\right]$.

4. In the setting of Problem 3, suppose that $\eta_0 = 1$ and

$$\xi_i = \begin{cases} \frac{1}{2} & \text{with probability } \frac{2}{3} \\ 2 & \text{with probability } \frac{1}{3}. \end{cases}$$

The process is stopped when $\eta_n = \frac{1}{4}$ or $\eta_n = 4$.

(a) Argue why the optional stopping theorem is applicable. No need to prove anything, argue by analogy with a standard problem.

(b) Compute the probabilities that the process is stopped at $\eta_n = \frac{1}{4}$ or $\eta_n = 4$, respectively.

(c) Compute the expected stopping time.
(Hint: use Problem 3(b).)

(d) Now suppose the process is only stopped when $\eta_n = \frac{1}{2}$. Is the optional stopping theorem still applicable? Explain!

(2+3+3+2)

Note that $\mu = E[\xi_i] = \frac{1}{2}\cdot\frac{2}{3} + 2\cdot\frac{1}{3} = 1$.

(a) This can be seen as a random walk on the lattice $2^n$, $n \in \mathbb{Z}$, so the same arguments as for the standard random walk apply. (Note that the stopping locations are part of the lattice, so the process cannot "jump the fence".)

(b) Optional stopping:

$$\underbrace{E[\eta_\tau]} = E[\eta_0] = 1$$

$$= \frac{1}{4}p + 4(1-p) \qquad p = P(\eta_\tau = \tfrac{1}{4})$$

$$\Rightarrow 4 - \frac{15}{4}p = 1 \quad \Rightarrow \quad p = \frac{4}{5}$$

(Problem 4 ctd.)

(c) Here, $\alpha = \left(\ln \frac{1}{2}\right) \frac{2}{3} + \left(\ln 2\right) \frac{1}{3} = -\frac{1}{3} \ln 2$

Optional stopping:

$$\underbrace{\mathbb{E}[\zeta_\tau]}_{=\mathbb{E}[\ln \eta_\tau - \alpha \tau]} = \mathbb{E}[\zeta_0] = 0$$

$$\Rightarrow \mathbb{E}[\tau] = \frac{1}{\alpha} \mathbb{E}[\ln \eta_\tau]$$

$$= \frac{1}{\alpha}\left(\left(\ln \frac{1}{4}\right) p + (\ln 4)(1-p)\right)$$

$$= -\frac{3}{\ln 2}\left(-2 \ln 2 \cdot \frac{4}{5} + 2 \ln 2 \cdot \frac{1}{5}\right)$$

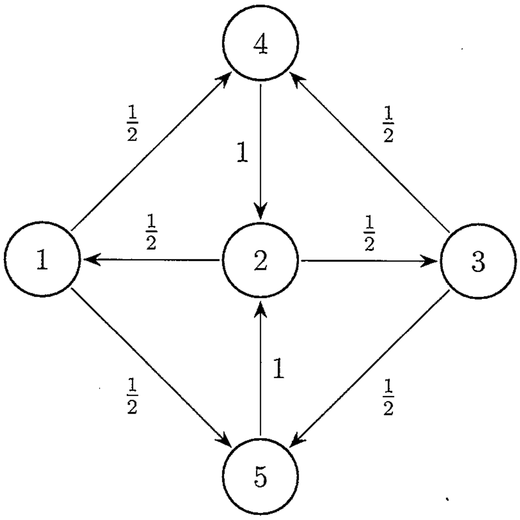
$$= -3\left(-\frac{8}{5} + \frac{2}{5}\right) = \frac{18}{5}$$

(d) If it were applicable, we should have

$$1 = \mathbb{E}[\eta_0] = \mathbb{E}[\eta_\tau] = \frac{1}{2} \quad \lightning$$

This situation is akin to "The Martingale".

5. Consider the following Markov chain.




(a) Write out the transition matrix.

(b) Identify the classes.

(c) Find the periods of all states.

(d) Find the stationary distribution(s).

(2+2+2+4)

(a)

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

(b) All states are interconnected, the state space is finite

$\Rightarrow$ there is only a single irreducible recurrent class

(c) All states have period 3.

(d) Need to solve $Px = x \Rightarrow (P-I)x = 0$

Use elementary row transformations:

$$P-I = \begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the invariant density is (normalized)

$$x = \frac{1}{6} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$